Lecture 2, Jan 14, 2022

Vector Spaces

- A *vector space* is the generalized concept of a vector that satisfies the usual rules of vector arithmetic
- Fundamental abstract operations addition $+$ and scalar multiplication $\cdot$ can be defined in any way, not just the common component-wise way
  - If it can be defined in any way, what makes a definition meaningful? When does it make sense?
- Definition: A *vector space* $\mathcal{V}$ over a field Γ of elements $\{\alpha, \beta, \gamma, \cdots\}$ called *scalars*, is a set of elements $\{u, v, w, \cdots\}$ such that the following *axioms* are satisfied:
  1. Vector addition denoted $u + v$ satisfies, for all $u, v, w \in \mathcal{V}$ (properties AI - AIV):
     1. Closure: $u + v \in \mathcal{V}$
     2. Associativity: $(u + v) + w = u + (v + w)$
     3. Existence of zero or null vector $0 \in \mathcal{V}$ such that $u + 0 = u$
     4. Existence of a negative or additive inverse $-u \in \mathcal{V}$ such that $u + (-u) = 0$
  2. Scalar multiplication denoted αu, such that for all $u, v \in \mathcal{V}$ and $\alpha, \beta \in \Gamma$ (properties MI - MIV):
     1. Closure: $\alpha u \in \mathcal{V}$
     2. Associativity: $\alpha(\beta u) = (\alpha\beta)u$
     3. Distributivity: $(\alpha + \beta)u = \alpha u + \beta u$, and $\alpha(u + v) = \alpha u + \alpha v$
     4. Unitary: For the identity $1 \in \Gamma$, $1u = u$
- Note that these properties imply commutativity for vector addition (will prove in a following lecture)
- A field Γ is a commutative group that has two operations, addition and multiplication (between scalars), and has a set of elements such that:
  1. Γ is commutative under addition
  2. Γ is commutative under multiplication excluding zero
  3. Multiplication is distributive over addition
- For us the field is almost always going to be $\mathbb{R}$; other examples of fields include the rationals, the complex numbers, etc
- A group is a set of elements $\{x, y, z, \cdots\}$ and a binary operation xy such that the operation is closed, associative, and there exists an inverse and identity for this operation; commutative groups additionally have $xy = yx$
- Matrices are an example of a vector space since they satisfy all of the requirements, so we can think of matrices as vectors
- Formally we would say $\mathcal{V}$ is a vector space over the field Γ under vector addition $+$ and scalar multiplication $\cdot$; as a shorthand we just say $\mathcal{V}$ is a vector space