Lecture 19, Mar 4, 2022

Linear Transformations and Matrices

- Not only do all matrices represent linear transformations, all linear transformations can be represented as a matrix; there exists a one-to-one relationship between matrices and linear transformations
- Consider $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$ and $\boldsymbol{w} \in \mathcal{W} = \mathscr{L}(\boldsymbol{v} \in \mathcal{V})$
  - $\boldsymbol{v} = \displaystyle\sum_{j=1}^{n} v_j \boldsymbol{e}_j$ given $E = \{ \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n \}$ is a basis for $\mathcal{V}$
  - $\boldsymbol{w} = \mathscr{L}\left(\displaystyle\sum_{j=1}^{n} v_j \boldsymbol{e}_j\right) = \displaystyle\sum_{j=1}^{n} v_j \mathscr{L}(\boldsymbol{e}_j)$
  - $\boldsymbol{w} = \displaystyle\sum_{i=1}^{m} w_i \boldsymbol{h}_i$ where $H = \{ \boldsymbol{h}_1, \cdots, \boldsymbol{h}_m \}$ is a basis for $\mathcal{W}$
  - $\mathscr{L}(\boldsymbol{e}_j) = \displaystyle\sum_{i=1}^{m} l_{ij} \boldsymbol{h}_i \implies \boldsymbol{w} = \sum_{j=1}^{n} v_j \mathscr{L}(\boldsymbol{e}_j) = \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} l_{ij} \boldsymbol{h}_i\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} l_{ij} v_j\right) \boldsymbol{h}_i$
  - Compare the 2 lines above, we get $w_i = \displaystyle\sum_{j=1}^{n} l_{ij} v_j$ which is a matrix multiplication: $\boldsymbol{w} = \boldsymbol{L}\boldsymbol{v}$
    - $\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \boldsymbol{L} = \begin{bmatrix} l_{ij} \end{bmatrix}$
  - The v_j are *coordinates* of $\boldsymbol{v}$ with respect to the basis E; w_i are coordinates of $\boldsymbol{w}$ with respect to the basis H
- There is a one-to-one relationship between $\boldsymbol{w} = \mathscr{L}(\boldsymbol{v})$ and $\boldsymbol{w} = \boldsymbol{L}\boldsymbol{v}$