Lecture 18, Mar 1, 2022

Linear Transformations/Operators

- Definition: A *linear transformation* is a transformation between two vector spaces $\mathcal{V}$ and $\mathcal{W}$, $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$ (where $\mathcal{V}$ is the *domain* and $\mathcal{W}$ is the *codomain*) that has the following properties:
  1. (L1) Distribution: $\mathscr{L}(\boldsymbol{u} + \boldsymbol{v}) = \mathscr{L}(\boldsymbol{u}) + \mathscr{L}(\boldsymbol{v}), \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$
  2. (L2) Homogeneity: $\mathscr{L}(\lambda \boldsymbol{v}) = \lambda \mathscr{L}(\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V}, \lambda \in \Gamma$
- These properties can be combined into one as $\mathscr{L}(\lambda \boldsymbol{u} + \mu \boldsymbol{v}) = \lambda \mathscr{L}(\boldsymbol{u}) + \mu \mathscr{L}(\boldsymbol{v})$
- A matrix $\boldsymbol{A} \in {}^m\mathbb{R}^n$ can be thought of as a linear transformation of $\boldsymbol{A} : {}^m\mathbb{R} \mapsto {}^n\mathbb{R}$
- The trace $\operatorname{tr} \boldsymbol{A} = \displaystyle\sum_{i=1}^{n} a_{ii}$ is a linear transformation $\operatorname{tr} : {}^n\mathbb{R}^n \mapsto \mathbb{R}$
  - L1: $\operatorname{tr}(\boldsymbol{A} + \boldsymbol{B}) = \displaystyle\sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr} \boldsymbol{A} + \operatorname{tr} \boldsymbol{B}$
  - L2: $\operatorname{tr}(\lambda \boldsymbol{A}) = \displaystyle\sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda \operatorname{tr} \boldsymbol{A}$
- Properties of linear transformations $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$:
  1. $\mathscr{L}(\boldsymbol{0}) = \boldsymbol{0}$
     - $\mathscr{L}(\boldsymbol{0}) = \mathscr{L}(0\boldsymbol{v}) = 0\mathscr{L}(\boldsymbol{v}) = \boldsymbol{0}$
  2. $\mathscr{L}(-\boldsymbol{v}) = -\mathscr{L}(\boldsymbol{v})$
     - $\mathscr{L}(-\boldsymbol{v}) = \mathscr{L}((-1)\boldsymbol{v}) = -1\mathscr{L}(\boldsymbol{v}) = -\mathscr{L}(\boldsymbol{v})$
  3. $\mathscr{L}\left(\displaystyle\sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i\right) = \sum_{i=1}^{n} \lambda \mathscr{L}(\boldsymbol{v}_i)$
     - Proven by induction
- Definition: The *image* of a linear transformation $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$ is $\operatorname{im} \mathscr{L} = \{\, \boldsymbol{w} \in \mathcal{W} \mid \boldsymbol{w} = \mathscr{L}(\boldsymbol{v}), \forall \boldsymbol{v} \in \mathcal{V} \,\}$ (column space for a matrix)
  - The image is a subspace of $\mathcal{W}$
- $\mathscr{L}$ maps $\mathcal{V}$ *into* $\mathcal{W}$ if $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$ but $\operatorname{im} \mathscr{L} \neq \mathcal{W}$
- $\mathscr{L}$ maps $\mathcal{V}$ *onto* $\mathcal{W}$ if $\operatorname{im} \mathscr{L} = \mathcal{W}$ (*surjective*), i.e. $\forall \boldsymbol{w} \in \mathcal{W}, \exists \boldsymbol{v} \ni \mathscr{L}(\boldsymbol{v}) = \boldsymbol{w}$
- $\mathscr{L}$ is *injective* if it is one-to-one, i.e. no two vectors in $\mathcal{V}$ maps onto the same vector in $\mathcal{W}$: $\nexists \boldsymbol{v}_1 = \boldsymbol{v}_2 \in \mathcal{V} \ni \mathscr{L}(\boldsymbol{v}_1) = \mathscr{L}(\boldsymbol{v}_2)$
- If $\mathscr{L}$ is surjective and injective then it is *bijective*
  - Bijective transformations have inverses
- Definition: The *kernel* of a linear transformation $\mathscr{L} : \mathcal{V} \mapsto \mathcal{W}$ is $\ker \mathscr{L} = \{\, \boldsymbol{v} \in \mathcal{V} \mid \mathscr{L}(\boldsymbol{v}) = \boldsymbol{0} \,\}$ (null space for a matrix)
  - The kernel is a subspace of $\mathcal{V}$
  - The kernel is everything in $\mathcal{V}$ that maps to $\boldsymbol{0}$ in $\mathcal{W}$
- The dimension formula for linear transformations: $\dim \ker \mathscr{L} + \dim \operatorname{im} \mathscr{L} = \dim \mathcal{V}$
  - Analogous to the dimension formula for matrices; for matrices $\ker \boldsymbol{A} = \operatorname{null} \boldsymbol{A}$, $\operatorname{im} \boldsymbol{A} = \operatorname{col} \boldsymbol{A}$ and $\dim \mathcal{V} = n$