Lecture 17, Feb 28, 2022

Non-Square Matrices

- Theorem IV: Let $\boldsymbol{A} \in {}^m\mathbb{R}^n$; then the following are equivalent:
  1. $\operatorname{rank} \boldsymbol{A} = n$
  2. The columns of $\boldsymbol{A}$ are linearly independent
  3. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$
  4. $\boldsymbol{A}^T\boldsymbol{A}$ is invertible
  5. $\boldsymbol{A}$ has a left inverse ($\exists \boldsymbol{B} \ni \boldsymbol{B}\boldsymbol{A} = \boldsymbol{1} \in {}^n\mathbb{R}^n$), where $\boldsymbol{B} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ is the left inverse, known as the Moore-Penrose pseudoinverse
- This necessitates $m \geq n$, i.e. $\boldsymbol{A}$ is a "tall" matrix, because if $n > m$ then the columns cannot be independent
- Proof:
  - $1 \implies 2$: $\operatorname{rank} \boldsymbol{A} = n \implies \dim \operatorname{col} \boldsymbol{A} = n$ so the columns are linearly independent as there are n columns
  - $2 \implies 3$: The columns are independent, so the only linear combination of the columns that add to 0 is all 0s, which is the zero vector
  - $3 \implies 4$: $\boldsymbol{A}^T\boldsymbol{A}$ is square, so it is invertible if and only if $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$
    * Lemma III: Let $\boldsymbol{s} \in {}^n\mathbb{R}$ and $\boldsymbol{s}^T\boldsymbol{s} = 0$ then $\boldsymbol{s} = \boldsymbol{0}$
      - Proof: $\boldsymbol{s}^T\boldsymbol{s} = \sum_{i=1}^{n} s_i^2 = 0$ but each $s_i^2 \geq 0$, which means all $s_i^2 = 0 \implies s_i = 0 \implies \boldsymbol{s} = \boldsymbol{0}$
    * $$\begin{aligned} &\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \\ \implies &\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0 \\ \implies &(\boldsymbol{A}\boldsymbol{x})^T(\boldsymbol{A}\boldsymbol{x}) = 0 \\ \implies &\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \\ \implies &\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$
  - $4 \implies 5$: $\boldsymbol{A}^T\boldsymbol{A}$ is invertible implies $\exists \boldsymbol{C} \ni \boldsymbol{C}\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{1}$; let $\boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}^T$, then $\boldsymbol{B}$ is the one-sided inverse
  - $5 \implies 1$: Show the columns are linearly independent: $$\begin{aligned} &\sum_{i=1}^{n} \boldsymbol{x}_i\boldsymbol{c}_i = \boldsymbol{0} \\ \implies &\boldsymbol{A}\boldsymbol{c} = \boldsymbol{0} \\ \implies &\boldsymbol{B}\boldsymbol{A}\boldsymbol{c} = \boldsymbol{0} \\ \implies &\boldsymbol{1}\boldsymbol{c} = \boldsymbol{0} \\ \implies &\boldsymbol{c} = \boldsymbol{0} \end{aligned}$$
- Theorem IV: Let $\boldsymbol{A} \in {}^m\mathbb{R}^n$ (this time $n \geq m$, i.e. $\boldsymbol{A}$ is short and wide); then the following are equivalent:
  1. $\operatorname{rank} \boldsymbol{A} = m$
  2. The rows of $\boldsymbol{A}$ are linearly independent
  3. $\boldsymbol{x}^T\boldsymbol{A} = \boldsymbol{0}^T \implies \boldsymbol{x} = \boldsymbol{0}$
  4. $\boldsymbol{A}\boldsymbol{A}^T$ is invertible
  5. $\boldsymbol{A}$ has a right inverse ($\exists \boldsymbol{B} \ni \boldsymbol{A}\boldsymbol{B} = \boldsymbol{1} \in {}^m\mathbb{R}^m$), where $\boldsymbol{B} = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}$ (also the Moore-Penrose pseudoinverse)