Lecture 16, Feb 18, 2022

The Dimension Formula

- Theorem II: The Dimension Formula: Let $\boldsymbol{A} \in {}^m\mathbb{R}^n$, then $\dim \operatorname{null} \boldsymbol{A} = n - \operatorname{rank} \boldsymbol{A}$ (or $\operatorname{rank} \boldsymbol{A} + \dim \operatorname{null} \boldsymbol{A} = n$)
  - Proof:
    * Let $\{ \boldsymbol{s}_1, \cdots, \boldsymbol{s}_k \}$ be a basis for $\operatorname{null} \boldsymbol{A}$; since $\operatorname{null} \boldsymbol{A} \sqsubseteq {}^n\mathbb{R}$, we can extend this to a basis for ${}^n\mathbb{R}$: $\{ \boldsymbol{s}_1, \cdots, \boldsymbol{s}_k, \boldsymbol{s}_{k+1}, \cdots, \boldsymbol{s}_n \}$
    * Claim: $\{ \boldsymbol{A}\boldsymbol{s}_{k+1}, \cdots, \boldsymbol{A}\boldsymbol{s}_n \}$ is a basis for $\operatorname{col} \boldsymbol{A}$ (if this is true, then $\dim \operatorname{col} \boldsymbol{A} = n - k = n - \dim \operatorname{null} \boldsymbol{A}$ and we're done)
      - Linear independence:

$$\begin{aligned} & \sum_{i=k+1}^{n} \lambda_i \boldsymbol{A}\boldsymbol{s}_i = \boldsymbol{0} \\ \implies & \boldsymbol{A}\left(\sum_{i=k+1}^{n} \lambda_i \boldsymbol{s}_i\right) = \boldsymbol{0} \\ \implies & \sum_{i=k+1}^{n} \lambda_i \boldsymbol{s}_i \in \operatorname{null} \boldsymbol{A} \\ \implies & \sum_{i=k+1}^{n} \lambda_i \boldsymbol{s}_i = \sum_{i=1}^{k} \mu_i \boldsymbol{s}_i \end{aligned}$$

        - Let $\mu_i = -\lambda_i \implies \sum_{i=1}^{n} \lambda_i \boldsymbol{s}_i = \boldsymbol{0} \implies \lambda_i = 0$ since $\{ s_i \}$ are a basis (note we can do this because i in the first summation and i in the second summation never have the same values)
      - Generation: $\operatorname{span}\{ \boldsymbol{A}\boldsymbol{s}_{k+1}, \cdots, \boldsymbol{A}\boldsymbol{s}_n \} \subseteq \operatorname{col} \boldsymbol{A}$ by Prop. I because each $\boldsymbol{s}_{k+1}, \cdots \boldsymbol{s}_n \in \operatorname{col} \boldsymbol{A}$
        - To go the other way:

$$\begin{aligned} & \boldsymbol{y} \in \operatorname{col} \boldsymbol{A} \\ \implies & \exists \boldsymbol{x} \ni \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{x} = \sum_{i=1}^{n} \beta_i \boldsymbol{s}_i \\ \implies & \boldsymbol{y} = \sum_{i=1}^{n} \beta_i \boldsymbol{A}\boldsymbol{s}_i \\ \implies & \sum_{i=k+1}^{n} \beta_i \boldsymbol{A}\boldsymbol{s}_i \\ \implies & \boldsymbol{y} \in \operatorname{span}\{ \boldsymbol{A}\boldsymbol{s}_{k+1}, \cdots, \boldsymbol{A}\boldsymbol{s}_n \} \end{aligned}$$

        - Therefore $\operatorname{col} A = \operatorname{span}\{ \boldsymbol{A}\boldsymbol{s}_{k+1}, \cdots, \boldsymbol{A}\boldsymbol{s}_n \}$
- Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$: there may exist no $\boldsymbol{x}$, or one unique $\boldsymbol{x}$, or infinitely many $\boldsymbol{x}$
  - No solution: $\boldsymbol{b} \notin \operatorname{col} \boldsymbol{A} \implies \operatorname{col} \boldsymbol{A} \subsetneq \operatorname{col}[\boldsymbol{A}|\boldsymbol{b}]$ or $\operatorname{rank} \boldsymbol{A} < \operatorname{rank}[\boldsymbol{A}|\boldsymbol{b}]$
  - Unique solution: $\boldsymbol{b} \in \operatorname{col} \boldsymbol{A}$ and $\operatorname{col} \boldsymbol{A} = \operatorname{col}[\boldsymbol{A}|\boldsymbol{b}]$ and $\operatorname{null} \boldsymbol{A} = \{ \boldsymbol{0} \}$
  - Infinite solutions: $\boldsymbol{b} \in \operatorname{col} \boldsymbol{A}$ and $\operatorname{col} \boldsymbol{A} = \operatorname{col}[\boldsymbol{A}|\boldsymbol{b}]$ and $\dim \operatorname{null} \boldsymbol{A} > 0$
- Theorem III: The following statements are equivalent for $\boldsymbol{A} \in {}^n\mathbb{R}^n$:
  1. $\boldsymbol{A}$ is invertible
  2. $\operatorname{rank} \boldsymbol{A} = n$ (i.e. $\boldsymbol{A}$ is full rank)
  3. $\boldsymbol{A}$ has linearly independent rows
  4. $\boldsymbol{A}$ has linearly independent columns
  5. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x} = \boldsymbol{0}$
  6. $\boldsymbol{z}^T\boldsymbol{A} = \boldsymbol{0}^T \implies \boldsymbol{z} = \boldsymbol{0}$
- Fredholm Alternative: Either $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution xor $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution
- Proof of Theorem III: In the case where we have a set of equivalent statements it's often most convenient to show a circular chain of implication, e.g. 1 implies 2 implies 3 implies 1
  - $1 \implies 2$: $\boldsymbol{A}$ is invertible means $\tilde{\boldsymbol{A}} = \boldsymbol{1} \implies \operatorname{rank} \boldsymbol{A} = \dim \operatorname{row} \boldsymbol{A} = \dim \operatorname{row} \tilde{\boldsymbol{A}} = n$

- $2 \implies 3$: $\operatorname{rank} \boldsymbol{A} = n \implies \dim \operatorname{row} \boldsymbol{A} = n$ but $\boldsymbol{A}$ only has n rows, so they have to be linearly independent
- $3 \implies 4$: Linearly independent rows $\implies \dim \operatorname{row} \boldsymbol{A} = n \implies \dim \operatorname{col} \boldsymbol{A} = n \implies$ the columns are linearly independent since there are n columns
- $4 \implies 5$: Linearly independent columns $\implies \sum_{i=1}^{n} x_i \boldsymbol{c}_i = \mathbf{0} \implies x_i = 0 \implies \boldsymbol{x} = \mathbf{0}$
  * Alternatively the independent columns implies $\operatorname{rank} \boldsymbol{A} = n \implies \dim \operatorname{null} \boldsymbol{A} = 0 \implies \{ \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x} = \mathbf{0} \}$
- $5 \implies 6$: $\{ \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x} = \mathbf{0} \} \implies \dim \operatorname{null} \boldsymbol{A} = 0 \implies \operatorname{rank} \boldsymbol{A} = n \implies \dim \operatorname{null} \boldsymbol{A}^T = 0 \implies \left\{ \boldsymbol{z}^T \boldsymbol{A} = \mathbf{0} \implies \boldsymbol{z} = \mathbf{0} \right\}$
- $6 \implies 1$: By contraposition, assume $\boldsymbol{A}$ is not invertible, which means there are zero rows in the rref, so $\operatorname{rank} \boldsymbol{A} < n$ so the rows are linearly dependent, which means $\boldsymbol{z}^T \boldsymbol{A} = \mathbf{0}$ has a nontrivial solution