Lecture 13, Feb 11, 2022

Null, Column, and Row Space

- The null space is defined as null $\boldsymbol{A} = \{\, \boldsymbol{x} \in {}^n\mathbb{R} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \,\} \subseteq \mathbb{R}$
- Consider $\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in {}^m\mathbb{R}^n$; $\boldsymbol{A}$ can be expressed as a set of rows $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix}, \boldsymbol{r}_k \in \mathbb{R}^n$ or a set of columns $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n \end{bmatrix}, \boldsymbol{c}_j \in {}^m\mathbb{R}$
- Define the *row space* of $\boldsymbol{A}$ as row $\boldsymbol{A} = \operatorname{span}\{\, \boldsymbol{r}_1, \cdots \boldsymbol{r}_m \,\} \sqsubseteq \mathbb{R}^n$, the *column space* of $\boldsymbol{A}$ as col $\boldsymbol{A} = \operatorname{span}\{\, \boldsymbol{c}_1, \cdots \boldsymbol{c}_n \,\} \sqsubseteq {}^m\mathbb{R}$
  - Both the row space and the column space have max dimension $\min\{\, m, n \,\}$ because they're restricted by the number of vectors in the spanning set and the space it's a subspace of
- The column space of $\boldsymbol{A}$ is equal to its image: col $\boldsymbol{A} = \{\, \boldsymbol{y} \in {}^m\mathbb{R} \mid \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \forall \boldsymbol{x} \in {}^n\mathbb{R} \,\}$
- Proposition I: Let $\boldsymbol{A} \in {}^m\mathbb{R}^n$ and $\boldsymbol{U} \in {}^m\mathbb{R}^m, \boldsymbol{V} \in {}^n\mathbb{R}^n$, then:
  1. row $\boldsymbol{U}\boldsymbol{A} \sqsubseteq$ row $\boldsymbol{A}$
     - All the rows of $\boldsymbol{U}\boldsymbol{A}$ are linear combinations of the rows of $\boldsymbol{A}$
  2. col $\boldsymbol{A}\boldsymbol{V} \sqsubseteq$ col $\boldsymbol{A}$
     - Similarly the columns of $\boldsymbol{A}\boldsymbol{V}$ are linear combinations of the columns of $\boldsymbol{A}$
  3. If $\boldsymbol{U}, \boldsymbol{V}$ are invertible, then row $\boldsymbol{U}\boldsymbol{A} =$ row $\boldsymbol{A}$ and col $\boldsymbol{A}\boldsymbol{V} =$ col $\boldsymbol{A}$
     - If $\boldsymbol{U}$ is invertible, consider $\boldsymbol{U} \to \boldsymbol{U}^{-1}$ and $\boldsymbol{A} \to \boldsymbol{U}\boldsymbol{A}$, so row $\boldsymbol{U}\boldsymbol{A} \sqsubseteq$ row $\boldsymbol{A} \iff$ row $\boldsymbol{U}^{-1}(\boldsymbol{U}\boldsymbol{A}) \sqsubseteq$ row $\boldsymbol{U}\boldsymbol{A} \implies$ row $\boldsymbol{A} \sqsubseteq$ row $\boldsymbol{U}\boldsymbol{A}$
     - Since the two subspaces are within each other they must be equal
- Proposition II: Let $\{\, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_r \,\} \subset {}^m\mathbb{R}, \boldsymbol{U} \in {}^m\mathbb{R}^m$ invertible, then $\{\, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_r \,\}$ is linearly independent iff $\{\, \boldsymbol{U}\boldsymbol{x}_1, \cdots, \boldsymbol{U}\boldsymbol{x}_r \,\}$ is linearly independent
  - Proof: $\sum_{i=1}^{n} \lambda_i(\boldsymbol{U}\boldsymbol{x}_i) = \boldsymbol{0} \iff \boldsymbol{U}\left(\sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i\right) = \boldsymbol{0} \iff \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i = \boldsymbol{0}$ so linearly independence of one set implies all $\lambda_i = 0$ which means the other set is linearly independent
  - We don't lose any information by multiplying a set of vectors by an invertible matrix