Lecture 12, Feb 8, 2022

Existence of Bases

- Theorem V: Let $\mathcal{V}$ be spanned by a finite set of vectors; then every linearly independent set in $\mathcal{V}$ can be extended to a basis for $\mathcal{V}$ (note we assume the set is not the zero set)
  - Proof by construction:
    1. Start with a linearly independent set $S_k = \{ \boldsymbol{v}_1, \cdots \boldsymbol{v}_k \} \subset \mathcal{V}$
    2. $\operatorname{span} S_k = \mathcal{V}$ or $\operatorname{span} S_k \neq \mathcal{V}$; if $\operatorname{span} S_k = \mathcal{V}$ then we have a linearly independent spanning set, which is a basis, so we're done
    3. Otherwise, $\exists(\boldsymbol{v}_{k+1} \in \mathcal{V}) \notin \operatorname{span} S_k$; by Theorem IV, $S_{k+1} = \{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1} \}$ is linearly independent
    4. If $\operatorname{span} S_k = \mathcal{V}$ then we have a basis; otherwise, repeat the previous step until we eventually get a basis
       * Because $\mathcal{V}$ is spanned by a finite set of vectors, this will always result in a basis
       * If it doesn't result in a basis then we'll end up with a set of linearly independent vectors that don't span $\mathcal{V}$ but has more vectors than the finite set that spans $\mathcal{V}$, which violates the fundamental theorem
- Theorem V gives a *maximally linearly independent set*
- Theorem VII: Let $\mathcal{V}$ be spanned by a finite set of vectors; then any spanning set for $\mathcal{V}$ can be reduced to a basis (i.e. it contains a basis)
  - Proof by construction:
    1. Start with a spanning set $\operatorname{span} S_p = \mathcal{V}$ where $S_p = \{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_p \} \subset \mathcal{V}$
    2. S_p is either linearly independent or not; if it is then S_p is a basis and we're done
    3. Otherwise, by Theorem I Corollary, $\exists \boldsymbol{v}_p \in S_p$ such that $\operatorname{span} S_{p-1} = \mathcal{V}$ where $S_{p-1} = \{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{p-1} \}$ (renumber the vectors such that v_p is that vector)
    4. If S_{p-1} is linearly independent then we have a basis; otherwise repeat the previous step until we eventually get a basis
       * This process must stop because eventually we get a set with just 1 vector which will be linearly independent
- Theorem VII gives a *minimally spanning set*
- Bases can be thought as minimally spanning sets or maximally independent sets
- Theorem VIII: Let $\mathcal{V}$ be such that $\dim \mathcal{V} = n$; then:
  1. Any linearly independent set of n vectors is a basis
  2. Any spanning set of n vectors is a basis
- Theorem VI: Let $\mathcal{U}, \mathcal{W} \sqsubseteq \mathcal{V}$, then:
  1. $\mathcal{U}, \mathcal{W}$ are finite dimensional with dimensions less than or equal to $\mathcal{V}$
  2. If $\mathcal{U} \subseteq \mathcal{W}$ then $\dim \mathcal{U} \leq \dim \mathcal{W}$
  3. $\mathcal{U} \subseteq \mathcal{W} \wedge \dim \mathcal{U} = \dim \mathcal{W} \implies \mathcal{U} = \mathcal{W}$