Lecture 11, Feb 7, 2022

Minimal Spanning Sets

- Theorem I: Let $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$. For every v_k (where $k = 1, 2, \cdots n$), $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots \boldsymbol{v}_n \} \subsetneq \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ iff $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ is linearly independent
  - i.e. if we have a set of linearly independent vectors and we take a vector out, the resulting span is always a strict subset (gets smaller)
  - Corollary (contrapositive): Let $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$, then for at least one v_k, $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots \boldsymbol{v}_n \}$ $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ if and only if $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ is linearly dependent
    - Note the original theorem is for every v_k but the contrapositive is for at least one v_k
  - Proof: At least one $\boldsymbol{v}_k$ implies linear dependence:
    - Let there be one $\boldsymbol{v}_k$ such that $\text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots \boldsymbol{v}_n \} = \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$
    - Then $\boldsymbol{v}_k \in \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots \boldsymbol{v}_n \}$
    - $\boldsymbol{v}_k = \sum_{\substack{i=1 \\ (i \neq k)}}^{n} \lambda_i \boldsymbol{v}_i \implies \lambda_1 \boldsymbol{v}_1 + \cdots + (-1)\boldsymbol{v}_k + \cdots + \lambda_n \boldsymbol{v}_n = \boldsymbol{0}$
    - Therefore the set $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n \}$ is not linearly independent because the coefficient on $\boldsymbol{v}_k$ can be nonzero
  - Proof: Linear dependence implies existence of $\boldsymbol{v}_k$:
    - $\sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$ has at least one $\lambda_k \neq 0$; take the vector associated with this λ to be $\boldsymbol{v}_k$
    - $\lambda_k \boldsymbol{v}_k = -\lambda_1 \boldsymbol{v}_1 - \cdots - \lambda_{k-1}\boldsymbol{v}_{k-1} - \lambda_{k+1}\boldsymbol{v}_{k+1} - \cdots - \lambda_n \boldsymbol{v}_n$
    - $\boldsymbol{v}_k = -\frac{\lambda_1}{\lambda_k}{}_1 \boldsymbol{v}_1 - \cdots - \frac{\lambda_{k-1}}{\lambda_k}\boldsymbol{v}_{k-1} - \frac{\lambda_{k+1}}{\lambda_k}\boldsymbol{v}_{k+1} - \cdots - \frac{\lambda_n}{\lambda_k}\boldsymbol{v}_n$
    - Therefore $\boldsymbol{v}_k \in \text{span}\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots \boldsymbol{v}_n \}$ so the span with $\boldsymbol{v}_k$ is the same as the span without $\boldsymbol{v}_k$
- Any minimum spanning set of a vector space is also a maximum independent set of that vector space
  - Taking any vector out of a set of linearly independent vectors loses information
- Theorem IV: Let $\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \} \subset \mathcal{V}$ be linearly independent; then for another $\boldsymbol{v} \in \mathcal{V}$, $\{ \boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$ is linearly independent if and only if $\boldsymbol{v} \notin \text{span}\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$; i.e. we can add a vector to a linearly independent set and keep it linearly independent if this vector is not already in the span
  - Contrapositive: If $\boldsymbol{v} \in \text{span}\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$ then $\{ \boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$ is linearly dependent
  - Proof:
    - $\boldsymbol{v} \in \text{span}\{ \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \} \implies \boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i \implies (-1)\boldsymbol{v} + \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n = \boldsymbol{0}$ therefore the set is linearly dependent
    - $\{ \boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$ linearly dependent means $\lambda \boldsymbol{v} + \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n = \boldsymbol{0}$ and not all λs equal zero
      - First we need to show $\lambda \neq 0$: If $\lambda = 0$, that means the rest of the λ_i have to be 0, which would mean $\{ \boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \}$ is linearly independent, creating a contradiction
      - Since $\lambda \neq 0$, $\boldsymbol{v} = -\frac{\lambda_1}{\lambda}\boldsymbol{v}_1 - \cdots - \frac{\lambda_n}{\lambda}\boldsymbol{v}_n$