Lecture 10, Feb 4, 2022

Basis Continued

- In general $\dim {}^n\mathbb{R} = \dim \mathbb{R}^n = n$ and $\dim {}^m\mathbb{R}^n = mn$
- The standard basis is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \cdots \right\}$
- Example: Consider the space of skew-symmetric matrices $\mathcal{U} = \left\{ \boldsymbol{S} \mid \boldsymbol{S} = -\boldsymbol{S}^T, \boldsymbol{S} \in {}^3\mathbb{R}^3 \right\}$
  - $\boldsymbol{S} = \boldsymbol{S}^T$ means the diagonal is forced to be zero
  - We can now write $\boldsymbol{S} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$
  - Since those 3 matrices span $\mathcal{U}$ and are linearly independent they form a basis, therefore $\dim \mathcal{U} = 3$
- Let $\mathcal{V}$ be a finite-dimensional vector space with $\dim \mathcal{V} = n$, then
  1. A linearly independent set of vectors in $\mathcal{V}$ can at most contain n vectors
  2. A spanning set for $\mathcal{V}$ must contain at least n vectors
- We can add vectors to any linearly independent set until we have $\mathcal{V} = n$ vectors; we can take away vectors from any spanning set until we have n vectors; at n vectors, we can have a spanning set that is linearly independent
  - Sometimes referred to as the rule of the extreme middle
- A basis characterizes a vector space