Lecture 9, Jan 31, 2022

Calculus with Parametric Curves

Tangents

- Since these curves can be more complex than functions there are more cases for tangents:
  1. Ordinary tangent
  2. No tangent (sharp point)
  3. Multiple tangents (due to the curve intersecting itself)
- To find the tangent we can again take the limit of a secant line
  - $m = \dfrac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} = \dfrac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}}$
  - Take the limit $h \to 0$ this just becomes $m = \dfrac{y'(t_0)}{x'(t_0)}$
    - Note we're differentiating with respect to t
  - The tangent line equation is $y'(t_0)(x - x_0) - x'(t_0)(y - y_0) = 0$
    - $\dfrac{y - y_0}{x - x_0} = \dfrac{y'(t_0)}{x'(t_0)} \implies y'(t_0)(x - x_0) = x'(t_0)(y - y_0) \implies y'(t_0)(x - x_0) - x'(t_0)(y - y_0) = 0$
- $x'(t_0) = 0$ gives a vertical tangent, $y'(t_0) = 0$ gives a horizontal tangent
  - If both are zero then we get no information
- Tangents can be used for curve sketching
  - Find derivatives of $x(t)$ and $y(t)$ and find locations of vertical and horizontal tangents
  - Also calculate slope at locations such as the origin and other points of interest

Areas

- Formula for area under parametric curve between $x(t_1)$ and $x(t_2)$ is just $A = \displaystyle\int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t$ (essentially a substitution)
- To calculate the area inside a closed curve, direction starts mattering; define the positive traversal direction such that the enclosed area is always on the left (i.e. counterclockwise)
- Going from t_1 to t_5 in the positive direction, the enclosed area is $A = -\displaystyle\int_{t_1}^{t_5} y(t)x'(t)\,\mathrm{d}t$, or $A = \displaystyle\int t_1{}^{t_5} x(t)y'(t)\,\mathrm{d}t$
  - Note the need for the negative sign when integrating with x but not y
  - Note the values of x and y at t_1 and t_5 are equal but $t_1 \neq t_5$
- Example: Ellipse $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$
  - Curve repeats $\theta \in [0, 2\pi]$
  - $x' = -a\sin\theta \implies A = -\displaystyle\int_0^{2\pi} -ab\sin^2\theta\,\mathrm{d}\theta = ab\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \pi ab$
  - We could also have done it with y' and get the same result

Arc Length

- Arc length is now given by $\sum\sqrt{\Delta x^2 + \Delta y^2}$; in the limit we get $s = \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$
  - Note if we let $x = t$, $y = f(t) = f(x)$ we get back the arc length formula for functions
- Example: $\begin{cases} x = \theta\cos\theta \\ y = \theta\sin\theta \end{cases}, \theta \in [0, 2\pi]$

- $s = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\,\mathrm{d}\theta$

$$= \int_0^{2\pi} \sqrt{1+\theta^2}\,\mathrm{d}\theta$$

$$= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\theta + \sqrt{1+\theta^2}\right|\right]_0^{2\pi}$$

$$= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right)$$

Surface Area

- Formula remains the same: $A = \int_a^b 2\pi y\,\mathrm{d}s$
  - Recall in parametric form $\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$