Lecture 8, Jan 28, 2022

Moments and Centres of Mass

- To find the centroid, make use of two principles:
  1. Symmetry: $(\bar{x}, \bar{y})$ must be on any axis of symmetry
     - In simple cases where we have multiple axes of symmetry we can just use the intersection of the axes to find the centroid
  2. Additivity: The centroid of a bigger piece is a weighted average of centroids of smaller pieces (where the weights are the areas of the smaller pieces)
     - $\bar{x} = \dfrac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \cdots}{A_1 + A_2 + \cdots}$ similarly for $\bar{y}$
- To extend this to a more complicated region we break any region into fine rectangles and take the limit to get an integral
  - Each piece has area $A_i = f(x_i^*)\Delta x_i$
  - $\bar{x}_i = x_i^*$
  - $\bar{y}_i = \dfrac{1}{2} f(x_i^*)$
  - Take the limit and $\bar{x} = \dfrac{\int_a^b x f(x)\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x}$, $\bar{y} = \dfrac{\frac{1}{2}\int_a^b f(x)^2\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x}$
- If we have a region bounded by two curves, use the additivity rule and subtract the smaller function
  - $\bar{x}A + \bar{x}_g A_g = \bar{x}_f A_f \implies \bar{x}A = \bar{x}_f A_f - \bar{x}_g A_g$
- Example: Region between $f(x) = 6, g(x) = 3$ between $x \in [2, 5]$
  - $\bar{x}A = \displaystyle\int_2^5 x(6-3)\,\mathrm{d}x = 3\left[\frac{1}{2}x^2\right]_2^5 = \frac{63}{2} \implies \bar{x} = \frac{7}{2}$
  - $\bar{y}A = \displaystyle\int_2^5 \frac{1}{2}(36-9)\,\mathrm{d}x = \frac{81}{2} \implies \bar{y} = \frac{9}{2}$
- Pappus' Theorem on Volumes: For a solid of revolution $V = 2\pi\bar{R}A$ where A is the area of the region being rotated and $\bar{R}$ is distance from the axis of rotation to the centroid of the region
  - Example: Elliptical torus with cross section as an ellipse with area $A = \pi ab$, radius is the major radius R so $V = 2\pi^2 Rab$
  - This theorem is equivalent to doing the integrals
  - Consider a washer method about x: $V_x = \displaystyle\int_a^b \pi\left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x = 2\pi\int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x = 2\pi\bar{y}A$
  - Consider the shell method about y: $V_y = \displaystyle\int_a^b 2\pi x(f(x) - g(x))\,\mathrm{d}x = 2\pi\int_a^b x(f(x) - g(x))\,\mathrm{d}x = 2\pi\bar{x}A$

Parametric Curves

- We can describe curves by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ where we introduced the parameter t and made it the new independent variable
- Example: Projectile motion in physics
- Example: Line between two points: $\begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \end{cases}$
- Parametric representations inherently contain more information; e.g. parameterizing projectile motion introduces information about time/velocity while $y(x)$ by itself only contains information about trajectory shape
- Intersection of two curves happens when $y_1(x) = y_2(x)$; collision happens when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$
  - To solve for collision, it can be helpful to solve for intersections first

- Example: $\begin{cases} x_1(t) = 2t + 6 \\ y_1(t) = 5 - 4t \end{cases}, \begin{cases} x_2(t) = 3 - 5\cos(\pi t) \\ y_2(t) = 1 + 5\sin(\pi t) \end{cases}, t \geq 0$
  - Intersection:
    * Curve 1: $t = \frac{x-6}{2} \implies y_1(x) = 17 - 2x, x \geq 6$
    * Curve 2: $(x-3)^2 + (y-1)^2 = 25$
    * Solving for intersections yields $(6, 5)$ and $(8, 1)$
  - Collision:
    * $(6, 5)$ happens on curve 1 at $t = 0$; curve 2 is at $(-2, 1)$ at this point, so there is no collision
    * $(8, 1)$ happens on curve 1 at $t = 1$; curve 2 is at $(8, 1)$ at this point, so there is a collision