Lecture 7, Jan 25, 2022

Arc Length

- Consider a curve $y = f(x)$ for $x \in [a, b]$, where $y'(x)$ exists; how do we define the length of the curve?
  - Approximate the curve by more and more finer straight line segments
  - With Δx between segments the length of each segment is $s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$
  - We may simplify this using the MVT; recall that $\dfrac{\Delta y_i}{\Delta x_i} = y'(x_i^*)$ for some $x_i^* \in [x_i, x_i + \Delta x_i]$ so $\Delta y_i = y'(x_i^*)\Delta x_i$
  - $s_i = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - Convert to integral for total length: $s = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$
- Example: $f(x) = x^{\frac{3}{2}}$ for $x \in [0, 44]$
  - $$f'(x) = \frac{3}{2}x^{\frac{1}{2}} \implies s = \int_0^{44} \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \left[\frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}x\right)^{\frac{1}{2}}\right]_0^{44} = 296$$
- Arc length integrals can be messy due to the square root; analytic solutions can only be found for a small set of functions

Surfaces of Revolution

- Side surface area of a cut off cone: $A = \pi(r + R)l$ where r is the smaller radius, R is the bigger radius and l is the slant height
- Rotate the curve $f(x)$ around the x axis to generate a surface; how do we find this area?
  - Cut into pieces, each piece is a cut off cone so the area is $A_i = \pi(f(x_{i-1}) + f(x_i))s_i = \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - Use IVT to simplify $f(x_{i-1}) + f(x_i)$: there exists some x_i^{**} such that $f(x_{i-1}) + f(x_i) = 2f(x_i^{**})$
  - $A_i = 2\pi f(x_i^{**})\sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - $A = \displaystyle\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x$
- Example: $f(x) = \sqrt{x}$ over $x \in [0, 1]$
  - $$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \implies A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,\mathrm{d}x = \pi\int_0^1 \sqrt{4x + 1}\,\mathrm{d}x = \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$
- Example: $f(x) = \dfrac{1}{x}$ over $x \in [1, \infty)$
  - Finite volume, infinite surface area (Gabriel's horn)
  - We can use a comparison test to demonstrate that this integral diverges

Applications to Physics and Engineering

Hydrostatic Pressure

- When an object is submerged in liquid it experiences a pressure force always perpendicular to the surface; the magnitude of force is $\rho g d$ where ρ is the density, g is the gravitational constant and d is the depth

- Consider a plate submerged vertically (x axis going down with 0 at the surface of the water)
  - $F_i = w(x_i^*)\Delta x_i \cdot \rho g x_i^*$ (area term times force term)
  - Taking the limit we get the integral $F = \displaystyle\int_a^b \rho g x w(x)\,\mathrm{d}x$
- Example: Circular water main with 1 meter radius; if we cap the pipe when its half filled with water, how much force will be pushing on the cap?
  - Since the pipe is half filled take the middle of the pipe as $x = 0$
  - Width of the pipe is $2\sqrt{1-x^2}$ by Pythagorean theorem
  - $$\begin{aligned} F &= 2\int_0^1 \rho g x\sqrt{1-x^2}\,\mathrm{d}x \\ &= 2\rho g\left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 \\ &= \frac{2}{3}\rho g \\ &= \frac{2}{3}\mathrm{m}^3 \cdot 1000\mathrm{kg/m}^3 \cdot 9.8\mathrm{m/s}^2 \\ &= 6533\mathrm{N} \end{aligned}$$