Lecture 6, Jan 24, 2022

Improper Integrals

- Remember: Infinity is NaN, so we must define any expression that contains it
- Definition: If $\lim_{b\to\infty} = \int_a^b f(x)\,\mathrm{d}x = L$, define $\int_0^\infty f(x)\,\mathrm{d}x = L$; these are called *improper integrals*
  - We can also have the lower limit go to infinity in the same way, or both bounds be infinite
  - Also define $[f(x)]_a^\infty$ as $\lim_{b\to\infty} [f(x)]_a^b$ if the limit exists/converges
  - If the limit doesn't exist then we say that the integral *diverges*
- Example: $$\begin{aligned}\int_0^\infty e^{-x}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b \\ &= \lim_{b\to\infty}(1-e^{-b}) \\ &= 1\end{aligned}$$
- Not all improper integrals converge! Example: $$\begin{aligned}\int_3^\infty \frac{1}{x}\,\mathrm{d}x &= \lim_{b\to\infty}[\ln x]_3^b \\ &= \lim_{b\to\infty}(\ln b - \ln 3) \\ &= \infty\end{aligned}$$
- Integrals can diverge for other reasons: $$\begin{aligned}\int_{-\infty}^{2\pi}\sin x\,\mathrm{d}x &= \lim_{a\to\infty}\int_a^{2\pi}\sin x\,\mathrm{d}x \\ &= \lim_{a\to\infty}[-\cos x]_a^{2\pi} \\ &= \lim_{a\to\infty}(-1+\cos a)\end{aligned}$$
  - Since $\cos a$ does not approach any value for $a\to\infty$ this integral is undefined
- General example: $\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ for $p>0, p\neq 1$ and $a>0$
  - $$\begin{aligned}\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x &= \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}x^{-p+1}\right]_a^b \\ &= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{1-p} - \frac{a^{-p+1}}{1-p}\right) \\ &= \frac{a^{1-p}}{p-1} && \text{for } p>1\end{aligned}$$
  - If $p<1$ then this will diverge
- Technique to demonstrate convergence: given f, g continuous and $0\leq f(x)\leq g(x)$ for $x\in[a,\infty]$, then if $\int_a^\infty g\,\mathrm{d}x$ converges so does $\int_a^\infty f\,\mathrm{d}x$; similarly if $\int_a^\infty f\,\mathrm{d}x$ diverges then so does $\int_0^\infty g\,\mathrm{d}x$
  - Example: $\int_2^\infty \frac{1}{\sqrt{1+x^{\frac{44}{17}}}}\,\mathrm{d}x$
    * We note $\frac{1}{\sqrt{1+x^{\frac{44}{17}}}} < \frac{1}{\sqrt{x^{\frac{44}{17}}}} = \frac{1}{x^{\frac{22}{17}}}$
    * Since $\frac{22}{17}>1$, $\int_2^\infty \frac{1}{x^{\frac{22}{17}}}\,\mathrm{d}x$ converges
    * Since this is larger than our integrand, our integral will also converge
  - Example: $\int_3^\infty \frac{1}{(7+x^2)^{\frac{1}{2}}}\,\mathrm{d}x$
    * Note $(7+x^2)^{\frac{1}{2}} < \sqrt{7}+x$ for $x\geq 3$

    - Therefore $\dfrac{1}{(7+x^2)^{\frac{1}{2}}} > \dfrac{1}{\sqrt{7}+x}$
    - Since $\displaystyle\int_3^\infty \frac{1}{\sqrt{7}+x}\,\mathrm{d}x$ diverges and our integrand is always greater than this integrand, our integral also diverges
- When we have both bounds infinite we can break it up: $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$
  - a can be anything here but we can usually choose it to be something convenient
  - Note $\displaystyle\int_{-\infty}^{\infty} \neq \lim_{b\to\infty}\int_{-b}^{b} f(x)\,\mathrm{d}x$
    - This works if the integral converges because in that case it doesn't matter how fast we approach infinity; however if the integral diverges this will give us the wrong answer
    - Example: $\displaystyle\int_{-\infty}^{\infty} x\,\mathrm{d}x$
      - If we break this up we can see this integral obviously doesn't converge
      - But if $\displaystyle\int_{-\infty}^{\infty} x\,\mathrm{d}x = \lim_{b\to\infty}\int_{-b}^{b} x\,\mathrm{d}x$

        $$= \lim_{b\to\infty}\left(\frac{b^2}{2} - \frac{b^2}{2}\right)$$

        $$= 0$$
      - We get zero because we happen to approach the two limits at the same rate
      - If instead $\displaystyle\lim_{b\to\infty}\int_{-b}^{2b} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{4b^2}{2} - \frac{b^2}{2}\right)$

        $$= \infty$$
        - If this integral was one-sided it wouldn't matter at what rate we approach infinity
- We can also have improper integrals where the interval contains a discontinuity
  - Suppose $\displaystyle\lim_{x\to b^-} f(x) = \infty$ then we can still define $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x$
- When the discontinuity is in the middle, break up the integral at the discontinuity; both pieces need to converge for the improper integral to converge
  - If we have a discontinuity at z, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to z^-}\int_a^c f(x)\,\mathrm{d}x + \lim_{c\to z^+}\int_c^b f(x)\,\mathrm{d}x$
  - We need to be careful because if we just plugged in the numbers as if there was no discontinuity we would get the wrong answer
  - Example: $\displaystyle\int_{-1}^{3} \frac{1}{x^2}\,\mathrm{d}x$
    - If we evaluate it as $\displaystyle\left[-\frac{1}{x}\right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$, which makes no sense as this integral should diverge and the function is always positive so we should never get a negative area
  - When we have an integral we need to make sure the integrand has no discontinuity over the region; if it does then we need to treat it as an improper integral
- For the interval 0 to 1, the $\dfrac{1}{x^p}$ rule is the reverse; if $p < 1$ then the integral converges, otherwise it diverges