Lecture 5, Jan 18, 2022

Partial Fraction Decomposition

- For $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of P is less than that of Q, we can write this as a sum of simpler polynomials
  - If the degree of P is equal to or greater than Q we must first perform long division and then decompose the remainder
- Factor Q into factors of $ax + b$ and $ax^2 + bx + c$ (linear and irreducible quadratic factors) and then express it as a sum of partial fractions: $\dfrac{A}{(ax+b)^i)}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^i}$
- Cases below:
  1. Q is a product of unique linear factors: $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{a_1x+b_1} + \dfrac{A_2}{a_2x+b_2} + \cdots$
     - Example: $\dfrac{x^2+2x+1}{2x^3+3x^2-2x} \to \dfrac{x^2+2x+1}{x(2x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2}$
       * $x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$
       * We can now expand the right side and group terms to obtain a system of equations for A, B, C
       * Alternatively we can set x to convenient values
         - $x = 0 \implies -1 = -2A \implies A = \dfrac{1}{2}$
         - $x = \dfrac{1}{2} \implies \dfrac{1}{4} = \dfrac{5}{4}B \implies B = \dfrac{1}{5}$
         - $x = -2 \implies -1 = 10C \implies C = -\dfrac{1}{10}$
  2. Q is a product of non-unique linear factors: If $(ax+b)^r$ occurs, then instead of just $\dfrac{A}{ax+b}$, we have $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_r}{(ax+b)^r}$
     - Example: $\dfrac{4x}{x^3-x^2-x+1} \to \dfrac{4x}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$
       * $4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$
       * We can once again expand and get the equations for the coefficients
       * We can still set convenient values of x:
         - $x = 1 \implies 4 = 2B \implies B = 2$
         - $x = -1 \implies -4 = 4C \implies C = -1$
         - Can't find an x that would cancel the other terms and leave A, but we can still obtain a relation for A
         - $x = 0 \implies 0 = -A + B + C \implies A = B + C = -1$
  3. Q contains unique irreducible quadratic factors: Each quadratic factor corresponds to a term of $\dfrac{Ax+B}{ax^2+bx+c}$ in the expansion
     - Example: $\dfrac{2x^2-x+4}{x^3+4x} \to \dfrac{2x^2-x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$
       * $2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$
       * Expand: $2x^2 - x + 4 = (A+B)x^2 + Cx + 4A \implies \begin{cases} A + B = 2 \\ C = -1 \\ 4A = 4 \end{cases} \implies \begin{cases} A = 1 \\ B = 1 \\ C = -1 \end{cases}$
       * Therefore $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{1}{x} + \dfrac{x-1}{x^2+4}$
     - To integrate the quadratic terms: $\displaystyle\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

- In general to integrate $\frac{Ax+B}{ax^2+bx+c}$, complete the square $\frac{Ax+B}{(ex+f)^2+g}$ then make the substitution $u = ex+f$ so the fraction is now $\frac{Cu+D}{u^2+g}$, which can be split into $\frac{Cu}{u^2+g}$ (can use substitution) and $\frac{D}{u^2+g}$ (can use the expression above)
- $$\begin{aligned}\int \frac{2x}{x^2+x+1}\,\mathrm{d}x &= \int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \frac{1}{x^2+x+1}\,\mathrm{d}x \\ &= \ln|x^2+x+1| + \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x \\ &= \ln|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C\end{aligned}$$

4. Q contains repeated irreducible quadratic factors: Similar to 2, each factor corresponds to terms of $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$
   - $\frac{Ax+B}{(x^2+\beta x+\gamma)^2} = \frac{A}{2}\left(\frac{2x+B}{(x^2+\beta x+\gamma)^2} + \frac{2\frac{B}{A}-B}{(x^2+\beta x+\gamma)^2}\right)$
     * The first can be directly solved using substitution, the second needs completing the square and then using trig substitution

- We can convert the irreducible quadratic terms into complex linear terms:
  - Example: $\int \frac{2x}{x^2+1}\,\mathrm{d}x$
    * Try $\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}$
    * $2x = A(x-i) + B(x+i)$
    * $x = i \implies 2i = 2iB \implies B = 1$ similarly $A = 1$
    * Integrating we get $\ln|x+i| + \ln|x-i| + C$ and the arguments of ln multiply to get us x^2+1 back
  - $\ln(a+ib) = \ln\sqrt{a^2+b^2} + i\tan^{-1}\left(\frac{b}{a}\right)$
    * Example: $\int \frac{1}{x^2+1}\,\mathrm{d}x$
- Nonrational functions can be made rational by an appropriate substitution
  - Example: $\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$
    * Let $u = \sqrt{x+4}$ then $u^2 = x+4 \implies x = u^2-4$ and $\mathrm{d}x = 2u\,\mathrm{d}u$ so $\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x = \int \frac{2u^2}{u^2-4}\,\mathrm{d}u = 2\int \left(1+\frac{4}{u^2-4}\right)\mathrm{d}u$
- Weierstrass substitution: $t = \tan\frac{x}{2}$
  - This allows us to convert ratios of sines and cosines to rational functions
  - $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$
  - $\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$

– Example:
$$\begin{aligned}\int \frac{1}{1+\cos x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int \frac{2}{(1+t^2)+(1-t^2)}\,\mathrm{d}t \\ &= \int \mathrm{d}t \\ &= t + C \\ &= \tan\left(\frac{x}{2}\right) + C\end{aligned}$$