Lecture 4, Jan 17, 2022

Trigonometric Integrals

Powers of Sines and Cosines

- When we have powers of sines and cosines, try to convert it to only one sin and the rest cos or one cos and the rest sin to u-sub
- Even powers can be converted using $\sin^2 x + \cos^2 x = 1$
  - Example: $$\begin{aligned}\int \cos^3 x \,\mathrm{d}x &= \int \cos^2 x \cos x \,\mathrm{d}x \\ &= \int (1 - \sin^2 x) \cos x \,\mathrm{d}x \\ &= \int (1 - u^2) \,\mathrm{d}u \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C\end{aligned}$$
- For odd powers, separate a single factor and convert the remaining even power
  - Example: $$\begin{aligned}\int \sin^5 x \cos^2 x \,\mathrm{d}x &= \int (1 - \cos^2 x)^2 \sin x \cos^2 x \,\mathrm{d}x \\ &= -\int (1 - u^2)^2 u^2 \,\mathrm{d}u \\ &= -\frac{u^3}{3} + 2\frac{2u^5}{5} - \frac{u^7}{7} + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$
- Half-angle identities: $$\begin{aligned}&\cos(2x) = \cos^2 x - \sin^2 x \text{ and } && \cos(2x) = \cos^2 x - \sin^2 x \\ \implies &\cos(2x) + 1 = 2\cos^2 x & \implies & 1 - \cos(2x) = 2\sin^2 x \\ \implies &\cos^2 x = \frac{1}{2}(1 + \cos(2x)) & \implies & \sin^2 x = \frac{1}{2}(1 - \cos(2x))\end{aligned}$$
  - Example: $$\begin{aligned}\int_0^\pi \sin^2 x \,\mathrm{d}x &= \frac{1}{2}\int_0^\pi 1 - \cos(2x) \,\mathrm{d}x \\ &= \left[\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)\right]_0^\pi \\ &= \frac{1}{2}\pi\end{aligned}$$
  - Example: $$\begin{aligned}\sin^4 x \,\mathrm{d}x &= \int (\sin^2 x)^2 \,\mathrm{d}x \\ &= \int \left(\frac{1}{2}(1 - \cos(2x))\right)^2 \,\mathrm{d}x \\ &= \frac{1}{4}\int \left(1 + \cos^2(2x) - 2\cos(2x)\right) \,\mathrm{d}x \\ &= \frac{1}{4}\int \left(1 + \frac{1}{2}(1 + \cos(4x)) - 2\cos(2x)\right) \,\mathrm{d}x \\ &= \frac{1}{4}\int \left(\frac{3}{2} + \frac{1}{2}\cos(4x) - 2\cos(2x)\right) \,\mathrm{d}x \\ &= \frac{1}{4}\left(\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C \\ &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\end{aligned}$$

- Remember $\sin x \cos x = \dfrac{1}{2}\sin(2x)$
- Summary of strategy for $\displaystyle\int \sin^m x \cos^n x \,\mathrm{d}x$:
  - If m is odd, save one sine factor and convert the rest to cosine; likewise for n
  - If both powers are even, use the half angle identities

Powers of Secant and Tangent

- Similar to sine and cosine since $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$; use identity $1 + \tan^2 x = \sec^2 x$
- Remember $\dfrac{\mathrm{d}}{\mathrm{d}x}\sec x = \tan x \sec x$
- Example: $$\begin{aligned}\int \tan^6 x \sec^4 x \,\mathrm{d}x &= \int \tan^6 x \sec^2 x (1 + \tan^2 x)\,\mathrm{d}x \\ &= \int u^6(1+u^2)\,\mathrm{d}u \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C \\ &= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C\end{aligned}$$
- Example: $$\begin{aligned}\int \tan^5 x \sec^7 x \,\mathrm{d}x = \int \tan^4 x \sec^6 x \sec x \tan x \,\mathrm{d}x &= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x \,\mathrm{d}x \\ &= \int (u^2-1)^2 u^6 \,\mathrm{d}u \\ &= \frac{u^1 1}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \\ &= \frac{1}{11}\sec^1 1x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C\end{aligned}$$
- Remember $\displaystyle\int \tan x \,\mathrm{d}x = \ln|\sec x| + C$ and $\displaystyle\int \sec x \,\mathrm{d}x = \ln|\sec x + \tan x| + C$
- For $\displaystyle\int \tan^m x \sec^n x \,\mathrm{d}x$:
  - For even power of sec, save one $\sec^2 x$, convert the rest to tangent, and substitute $u = \tan x$
  - For odd power of tan, save one tan for a $\tan x \sec x$, cover the rest of $\tan^2$ to secant, then substitute $u = \sec x \implies \mathrm{d}u = \tan x \sec x \,\mathrm{d}x$
- We may have to integrate by parts; example: $$\begin{aligned}\int \sec^3 x \,\mathrm{d}x &= \sec x \tan x - \int (\sec x \tan x)\tan x \,\mathrm{d}x \\ &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,\mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \,\mathrm{d}x + \ln|\sec x + \tan x| + C\end{aligned}$$
  - Now we can solve for $\displaystyle\int \sec^3 x \,\mathrm{d}x$ since it appears on both sides

Product-To-Sum

- Integrals of the form $\displaystyle\int \sin(mx)\cos(nx)\,\mathrm{d}x$, $\displaystyle\int \sin(mx)\sin(nx)\,\mathrm{d}x$ or $\displaystyle\int \cos(mx)\cos(nx)\,\mathrm{d}x$ can be solved using the product-to-sum identities
- $\sin A \sin B = \dfrac{1}{2}(\cos(A-B) - \cos(A+B))$
- $\cos A \cos B = \dfrac{1}{2}(\cos(A-B) + \cos(A+B))$
- $\sin A \cos B = \dfrac{1}{2}(\sin(A-B) + \sin(A+B))$

Trigonometric Substitution

- Inverse substituting x for a trig function could simplify radicals of some forms
- $\sqrt{a^2 - x^2}$: Substitute $x = a\sin\theta \implies \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$
$$= a\sqrt{1 - \sin^2\theta}$$
$$= a|\cos\theta|$$
- $\sqrt{a^2 + x^2}$: Substitute $x = a\tan\theta \implies \sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta}$
$$= a\sqrt{1 + \tan^2\theta}$$
$$= a|\sec\theta|$$
- $\sqrt{x^2 - a^2}$: Substitute $x = a\sec\theta \implies \sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta + a^2}$
$$= a\sqrt{\sec^2\theta - 1}$$
$$= a|\tan\theta|$$
    - Alternatively, use the hyperbolic substitution $x = \cosh^2 t \implies \sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)}$
$$= a\sinh t$$
- Example: $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\,\mathrm{d}x = \int \frac{3\cos\theta}{9\sin^2\theta} 2\cos\theta\,\mathrm{d}\theta$
$$= \int \cot^2\theta\,\mathrm{d}\theta$$
$$= \int (\csc^2\theta - 1)\,\mathrm{d}\theta$$
$$= -\cot\theta - \theta + C$$
    - To return this to x, $x = 3\sin\theta \implies \sin\theta = \dfrac{x}{3} \implies \theta = \sin^{-1}\left(\dfrac{x}{3}\right)$; draw a triangle with the right sides and we get $\cot\theta = \dfrac{\sqrt{9 - x^2}}{x}$
    - For definite integrals we can just change the limits of integration to avoid having to convert back
- Example: Find the area of the ellipse $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$
    - By symmetry we only need to worry about one quadrant where $x, y \geq 0$, so solving for y we have $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$
    - Substitute $x = a\sin\theta \implies \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$; change the bounds of integration: $x = a\sin\theta = 0 \implies \theta = 0, x = a\sin\theta = a \implies \sin\theta = 1 \implies \theta = \dfrac{\pi}{2}$
    - $A = 4\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\mathrm{d}x = 4\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\,\mathrm{d}\theta$
$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\ \mathrm{d}\theta$$
$$= 4ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2\theta))\,\mathrm{d}\theta$$
$$= 2ab\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$
$$= \pi ab$$