Lecture 36, Apr 11, 2022

Differentiability of an Integral w.r.t. Its Parameter

- Consider $F(x) = \int_c^d f(x,y)\,\mathrm{d}y$; what happens when we try to take its derivative?
- Theorem: If $f(x,y)$ has a continuous derivative with respect to x in the closed rectangle $x \in [a,b], y \in [c,d]$, then for $x \in [a,b]$, $\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \frac{\partial f}{\partial x}\,\mathrm{d}y$
  - Proof: Given $x, x+h \in [a,b]$, then:
    * $\begin{aligned}\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\int_c^d f(x+h,y)\,\mathrm{d}y - \frac{1}{h}\int_c^d f(x,y)\,\mathrm{d}y \\ &= \frac{1}{h}\int_c^d f(x+h,y) - f(x,y)\,\mathrm{d}y\end{aligned}$
    * $\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\int_c^d \frac{f(x+h,y)-f(x,y)}{h}\,\mathrm{d}y \\ &= \int_c^d \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\,\mathrm{d}y \\ &= \int_c^d \frac{\partial f}{\partial x}\,\mathrm{d}y\end{aligned}$
    * We can bring the limit inside the integral, because the integral is a sum and limit laws allow us to distribute the limit over sums
- Example: $F(x) = \int_2^4 e^{xy}\,\mathrm{d}y$
  - Doing the integral first: $F(x) = \left[\frac{e^{xy}}{x}\right]_2^4 = \frac{e^{4x}-e^{2x}}{x}$
    * $\frac{\mathrm{d}F}{\mathrm{d}x} = e^{4x}\left(\frac{4x-1}{x^2}\right) - e^{2x}\left(\frac{2x-1}{x^2}\right)$
  - Using the theorem: $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_2^4 \frac{\partial}{\partial x}e^{xy}\,\mathrm{d}x = \int_2^4 ye^{xy}\,\mathrm{d}y$
    * Using integration by parts: $\left[\frac{y}{x}e^{xy}\right]_2^4 - \int_2^4 \frac{e^{xy}}{x}\,\mathrm{d}y = \left[\left(\frac{y}{x} - \frac{1}{x^2}\right)e^{xy}\right]_2^4 = e^{4x}\left(\frac{4x-1}{x^2}\right) - e^{2x}\left(\frac{2x-1}{x^2}\right)$
- Consider $A(t) = \int_{x_1(t)}^{x_2(t)} f(x)\,\mathrm{d}x$ and note $\frac{\mathrm{d}A}{\mathrm{d}t} = f(x_2(t))\frac{\mathrm{d}x_2}{\mathrm{d}t} - f(x_1(t))\frac{\mathrm{d}x_1}{\mathrm{d}t}$ by the chain rule and FTC
- Theorem: Leibniz's Rule: Given a region R in the xy plane in which $\phi_1(x), \phi_2(x)$ and $f(x,y)$ are continuously differentiable, if $F(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y$, then $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}\,\mathrm{d}y + f(x,\phi_2(x))\frac{\mathrm{d}\phi_2}{\mathrm{d}x} - f(x,\phi_1(x))\frac{\mathrm{d}\phi_1}{\mathrm{d}x}$
- Example: $F(x) = \int_0^{x^2} \sin(xy)\,\mathrm{d}y$
  - $f(x,y) = \sin(xy), \phi_1(x) = 0, \phi_2(x) = x^2$
  - $\frac{\partial f}{\partial x} = y\cos(xy), \frac{\mathrm{d}\phi_1}{\mathrm{d}x} = 0, \frac{\mathrm{d}\phi_2}{\mathrm{d}x} = 2x$
  - $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_0^{x^2} y\cos(xy)\,\mathrm{d}y + 2x\sin(x\cdot x^2) + 0\sin(0) = \int_0^{x^2} y\cos(xy)\,\mathrm{d}y + 2x\sin(x^3)$
- This can be used as an integration technique (Feynman integration)

- Example: $F(x) = \int_0^1 \frac{y^x - 1}{\ln y} \, \mathrm{d}y$ for $x > -1$
  - $$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}x} &= \int_0^1 \frac{\partial}{\partial x} \left(\frac{y^x - 1}{\ln y} \right) \, \mathrm{d}y \\ &= \int_0^1 \frac{y^x \ln y}{\ln y} \, \mathrm{d}y \\ &= \int_0^1 y^x \, \mathrm{d}y \\ &= \left[\frac{y^{x+1}}{x+1} \right]_0^1 \\ &= \frac{1}{x+1} \end{aligned}$$
  - $F(x) = \int \frac{1}{x+1} \, \mathrm{d}x = \ln|x+1| + C$
  - From the original expression $F(0) = \int_0^1 \frac{y^0 - 1}{\ln y} \, \mathrm{d}y = 0 \implies C = 0 \implies F(x) = \ln(x+1)$