Lecture 35, Apr 8, 2022

Multivariable Optimization Example: Rocket Science

- Tsiolkovsky's Rocket Equation: $\Delta v = I_{sp} \ln \frac{m_i}{m_f}$
  - Δv is the momentum change given to the rocket
  - I_{sp} is the specific impulse of the fuel (i.e. speed of exhaust gases)
  - m_i and m_f are the initial and final mass
- Escape velocity of the Earth is about $\Delta v \approx 11$km/s; $I_{sp} \approx 3.2$km/s for common rocket fuel
- For a single stage rocket: $\frac{m_i}{m_f} = \exp\left(\frac{11}{3.2}\right) = 31.1 \implies \frac{m_{fuel}}{m_i} = \frac{m_i - m_f}{m_i} = 1 - \frac{m_f}{m_i} = 0.97$
  - 97% of the rocket would have to be fuel for the rocket to break escape velocity
  - Comparison: Aluminum beverage can is 96%
- What if we limit fuel to be 80% of the total weight?
- Break down $m_i = A + M_s + M_f$ where A is the payload, M_s is the rocket structure, M_{fuel} is the fuel
  - Define $S = \frac{M_s}{M_s + M_{fuel}}$ as the structure factor, i.e. the fraction of the mass without payload that's the structure
  - Define rocket mass $M_r = M_s + M_{fuel}$
  - $\frac{m_i}{m_f} = \frac{A + M_s + M_{fuel}}{A + M_s} = \frac{A + M_r}{A + SM_r} \implies \Delta v = I_{sp} \ln\left(\frac{A + M_r}{A + SM_r}\right)$
    - If we plug in $S = 0.2$ and I_{sp} from before, without any payload this would give $\Delta v = 5.15$km/s
- Given a 3-stage rocket, how can we optimize the sizes of the 3 stages to minimize the total rocket mass?
  - Let the 3 stages have masses M_1, M_2, M_3 and the same structure factor S
  - The first stage has initial mass $A + M_1 + M_2 + M_3$; after the fuel is expended, $M_1 \to SM_1$, then the second stage has mass $A + M_2 + M_3$ and so on
  - $\Delta v_{tot} = \Delta v_1 + \Delta v_2 + \Delta v_3$

$$= I_{sp}\left[\ln\left(\frac{A + M_1 + M_2 + M_3}{A + SM_1 + M_2 + M_3}\right) + \ln\left(\frac{A + M_2 + M_3}{A + SM_2 + M_3}\right) + \left(\frac{A + M_3}{A + SM_3}\right)\right]$$

- Problem: Given A, Δv, I_{sp}, and S for a 3 stage rocket, find M_1, M_2, M_3 such that $M_r = f(M_1, M_2, M_3) = M_1 + M_2 + M_3$ is minimized, subject to the constraint of the rocket equation above
  - Constraint: $g(M_1, M_2, M_3) = \ln\left(\frac{A + M_1 + M_2 + M_3}{A + SM_1 + M_2 + M_3}\right) + \ln\left(\frac{A + M_2 + M_3}{A + SM_2 + M_3}\right) + \left(\frac{A + M_3}{A + SM_3}\right) = \frac{\Delta v}{I_{sp}}$
  - $\nabla f = (1, 1, 1)$
  - $g = \ln(A + M_1 + M_2 + M_3) - \ln(A + SM_1 + M_2 + M_3) + \ln(A + M_2 + M_3) - \ln(A + SM_2 + M_3) + \ln(A + M_3) - \ln(A + SM_3)$
  - For $M_1 : \lambda \frac{\partial f}{\partial M_1} = \frac{\partial g}{\partial M_1} \implies \lambda = \frac{1}{A + M_1 + M_2 + M_3} - \frac{S}{A + SM_1 + M_2 + M_3}$
  - For $M_2 : \lambda \frac{\partial f}{\partial M_2} = \frac{\partial g}{\partial M_2} \implies \lambda = \frac{1}{A + M_1 + M_2 + M_3} - \frac{1}{A + SM_1 + M_2 + M_3} + \frac{1}{A + M_2 + M_3} - \frac{S}{A + SM_2 + M_3}$
  - For $M_3 : \lambda \frac{\partial f}{\partial M_3} = \frac{\partial g}{\partial M_3} \implies \lambda = \frac{1}{A + M_1 + M_2 + M_3} - \frac{1}{A + SM_1 + M_2 + M_3} + \frac{1}{A + M_2 + M_3} - \frac{1}{A + SM_2 + M_3} + \frac{1}{A + M_3} - \frac{S}{A + SM_3}$
  - Subtracting the third from the second equation gives: $0 = \frac{1}{A + SM_2 + M_3} - \frac{S}{A + SM_2 + M_3} + \frac{S}{A + SM_3} - \frac{1}{A + M_3} \implies -\frac{S}{A + SM_3} = \frac{1 - S}{A + SM_2 + M_3} - \frac{1}{A + M_3}$
    - $-\frac{S(A + M_3)}{A + SM_3} = \frac{(1 - S)(A + M_3)}{A + SM_2 + M_3} - 1 = \frac{A + M_3 - SA - SM_3 - A - SM_2 - M_3}{A + SM_2 + M_3} =$

$-\dfrac{S(A+M_2+M_3)}{A+SM_2+M_3}$

* $\dfrac{A+M_3}{A+SM_3} = \dfrac{A+M_2+M_3}{A+SM_2+M_3}$
* Note these are the things inside ln in the second and third terms in g

– Subtracting the second equation from the first and going through the same steps, we obtain the relation $\dfrac{A+M_2+M_3}{A+SM_2+M_3} + \dfrac{A+M_1+M_2+M_3}{A+SM_1+M_2+M_3}$

* These are the arguments in ln in the first and second terms in g

– This basically tells us that $\dfrac{m_i}{m_f}$ should be the same for each stage of the rocket

– Let $N = \dfrac{A+M_3}{A+SM_3} \implies \dfrac{\Delta v}{I_{sp}} = 3\ln N \implies N = \exp\left(\dfrac{\Delta v}{3I_{sp}}\right)$

– $M_3 = A\dfrac{N-1}{1-SN}, M_2 = (A+M_3)\dfrac{N-1}{1-SN}, M_1 = (A+M_2+M_3)\dfrac{N-1}{1-SN}$