Lecture 34, Apr 5, 2022

Lagrange Multipliers

- In general the goal is to maximize or minimize $f(x, y)$ subject to a constraint of $g(x, y) = k$
  - Geometrically, picture the level curves of $f(x, y)$ along with the curve $g(x, y) = k$
  - A solution must lie on both $g(x, y) = k$ and one of the level curves of $f(x, y)$; the goal is to find the largest c such that $f(x, y) = c$ intersects $g(x, y) = k$
  - This happens when the two curves just touch each other at a single point, i.e. they're tangent to each other
    * Note if they crossed, there would always be a way to choose a larger or smaller c such that they touch at a single point
- Since the curves are tangent, they share the same tangent and thus $\nabla g \parallel \nabla f$ or $\nabla f = \lambda \nabla g$; λ is the *Lagrange multiplier*
- In the 2D case, we need to solve $\begin{cases} g(x_0, y_0) = k \\ f_x(x_0, y_0) = \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0) \end{cases}$
  - Together we have 3 equations and 3 unknowns x_0, y_0, λ, so we can solve the system
  - Often we don't care about finding λ, only x_0, y_0
- In 3D, the surfaces are tangent and share the same tangent plane so again the gradients are parallel
  - In general $\begin{cases} g(\vec{x}_0) = k \\ \nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0) \end{cases}$
- Example: $f(x, y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$
  - $\nabla f = (2x, -2y), \nabla g = (2x, 2y)$
  - $\begin{cases} x_0^2 + y_0^2 = 1 \\ 2x_0 = \lambda 2x_0 \\ -2y_0 = \lambda 2y_0 \end{cases}$
  - From the second equation, either $x_0 = 0$ or $\lambda = 1$; from the third equation either $y_0 = 0$ or $\lambda = -1$
  - Cases:
    1. $\lambda = 1, y_0 = 0 \implies x_0 = \pm 1$
    2. $\lambda = -1, x_0 = 0 \implies y_0 = \pm 1$
- Note Lagrange's method doesn't tell us whether we have a max or min, but it does give us all the max/min
- Example: Maximize $f(x, y, z) = xyz$ subject to $x^3 + y^3 + z^3 = 1, x, y, z \geq 0$
  - $\begin{cases} x^3 + y^3 + z^3 = 1 \\ yz = \lambda 3x^2 \\ xz = \lambda 3y^2 \\ zy = \lambda 3z^2 \end{cases} \implies \begin{cases} xyz = \lambda 3x^2 \\ xyz = \lambda 3y^2 \\ xyz = \lambda 3z^2 \end{cases}$
  - $\lambda x^3 = \lambda y^3 = \lambda z^3$
  - Eliminate $\lambda = 0$ possibility because if $\lambda = 0$, $x = y = z = 0$ which would be a minimum
  - $x^3 = y^3 = z^3 \implies x = y = z$
  - Plugging this back into our constraint we get $x = y = z = \sqrt[3]{\frac{1}{3}} \implies f(x, y, z) = \frac{1}{3}$
- Problems of this type are easy to set up, but solving the system of equations is complicated

Two Constraints Problem

- Maximize or minimize $f(x, y, z)$ subject to $g(x, y, z) = k$ and $h(x, y, z) = c$
  - Geometrically we're trying to maximize or minimize f on the intersection between g and h
  - The 3D surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$ intersect at a curve
- Note that since the gradient is normal to a level surface, $\vec{T}$ for the intersection curve is normal to both ∇h and ∇g

- Therefore $\vec{T} = \nabla h \times \nabla g$
- By the same logic as before, $\vec{T}$ must be in the tangent plane of f at the max/min, so ∇f is perpendicular to $\vec{T}$
  - Since $\nabla f \perp \vec{T}$, it must be in the plane defined by ∇g and ∇h since that plane is also perpendicular to $\vec{T}$
  - $\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0) + \mu \nabla h(\vec{x}_0)$ is our new equation
- The problem is now reduced to a set of 5 equations
- Example: $f(x, y, z) = xy + 2z$ on a circle of intersection between the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$
  - $\nabla f = (y, x, 2), \nabla g = (1, 1, 1), \nabla h = (2x, 2y, 2z)$
  - $\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 = 24 \\ y = \lambda + \mu \cdot 2x \\ x = \lambda + \mu \cdot 2y \\ 2 = \lambda + \mu \cdot 2z \end{cases}$
  - $(x - y) = 2\mu(y - x) \implies (x - y)(1 + 2\mu) = 0 \implies y = x \text{ or } \mu = -\frac{1}{2}$
  - Cases:
    1. $x = y \implies 2x + z = 0 \implies z = -2x \implies x^2 + x^2 + (-2x)^2 = 24 \implies 6x^2 = 24 \implies x = \pm 2, y = \pm 2, z = \mp 4$
       * This produces the points $f(2, 2, -4) = -4$ and $f(-2, -2, 4) = 12$
    2. $\mu = -\frac{1}{2} \implies \begin{cases} x = \lambda - y \\ 2 = \lambda - z \end{cases} \implies x + y = 2 + z \implies 2 + z + z = 0 \implies z = -1, x + y = 1 \implies$ $\begin{cases} x^2 + y^2 = 24 - z^2 = 23 \\ (x + y)^2 = x^2 + y^2 + 2xy = 1^2 = 1 \end{cases} \implies xy = -11 \implies y = 1 - x \implies x(1 - x) = -11 \implies x^2 - x - 11 = 0 \implies x = \frac{1 \pm 3\sqrt{5}}{2}, y = \frac{1 - 3\sqrt{5}}{2}$
       * This produces the points $f\left(\frac{1 + 3\sqrt{5}}{2}, \frac{1 - 3\sqrt{5}}{2}, 1\right) = -13$ and $f\left(\frac{1 - 3\sqrt{5}}{2}, \frac{1 + 3\sqrt{5}}{2}, 1\right) = -13$

Reconstructing a Function from its Gradient

- If we have ∇f, how do we obtain f?
- Method 1: Integrate one of the partial derivatives, creating a "constant of integration" that's a function of the other variables; take the partial derivative with respect to the other variables, and compare against the gradient to solve for the constants of integration
- Example: $\nabla f = (1 + y^2 + xy^2)\hat{\imath} + (x^2 y + y + 2xy + 1)\hat{\jmath}$
  - $\frac{\partial f}{\partial x} = 1 + y^2 + xy^2 \implies f = x + xy^2 + \frac{1}{2}x^2 y^2 + \phi(y)$
    * Note the constant of integration here is a "constant" with respect to x only, meaning it could be any function of y
  - Now differentiate: $\frac{\partial f}{\partial y} = 2xy + x^2 y + \phi'(y) = x^2 y + y + 2xy + 1 \implies \phi'(y) = y + 1 \implies \phi(y) = \frac{1}{2}y^2 + y + C$
  - Therefore $f(x, y) = x + xy^2 + \frac{1}{2}x^2 y + \frac{1}{2}y^2 + y + C$
- Method 2: Integrate all partial derivatives, and match the terms to get the final expression
- Example: $\nabla f = (\cos x - y \sin x)\hat{\imath} + (\cos x + z^2)\hat{\jmath} + (2yz)\hat{k}$
  - $f_x = \cos x - y \sin x \implies f = \sin x + y \cos x + \phi_1(y, z)$
  - $f_y = \cos x + z^2 \implies f = y \cos x + yz^2 + \phi_2(x, z)$
  - $f_z = 2yz \implies f = yz^2 + \phi_3(x, y)$

  - Since all 3 of these have to be true, we can conclude that $f(x,y,z) = \sin x + y\cos x + yz^2 + C$
- Not all $P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ are gradients!
  - Example: $\nabla f(x,y) = y\hat{\imath} - x\hat{\jmath}$
    * $f_x = y \implies f_{xy} = 1$
    * $f_y = -x \implies f_{yx} = -1$
    * $\dfrac{\partial^2 f}{\partial y \partial x} \neq \dfrac{\partial^2 f}{\partial x \partial y}$
    * Since mixed partials do not agree, but all derivatives are continuous, this contradicts Clairaut's theorem so f could not exist
- Theorem: Let P and Q be functions of two variables, each continuous and differentiable, then $P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is a gradient iff $\dfrac{\partial P}{\partial y}(x,y) = \dfrac{\partial Q}{\partial x}(x,y)$
  - In 3 dimensions, we need to apply this 3 times (comparing $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial^2 f}{\partial z \partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x}$), but there's no need for the third order partials