Lecture 33, Apr 4, 2022

Absolute Extrema

- As with single variable optimization, the absolute min/max can occur at a local min/max or on the boundary (which are curves in the multivariable case)
- Theorem: If f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set (extreme value in multiple variables)
- First find critical points inside the region where the gradient is zero or DNE, and then find critical points along the boundary, and then the end points
  - By parameterizing the boundary curve, we can find its derivative and find when that equals zero
  - Either use the chain rule or substitute back into the original function
- Example: $f(x, y) = (x-4)^2 + y^2$ on $\{ (x, y) : 0 \leq x \leq 2, x^3 \leq y \leq 4x \}$
  - Critical points: $\nabla f = (2(x-4), 2y) = \vec{0} \implies x = 4, y = 0$
    - The critical point is not in the set we're looking at, which means both absolute max and min are on the boundary
  - Boundary 1: $y = x^3, 0 \leq x \leq 2$
    - Parameterize the boundary $\begin{cases} x = t \\ y = t^3 \end{cases}, 0 \leq t \leq 2 \implies \vec{r}_1(t) = t\hat{i} + t^3\hat{j}$
    - Now we have a single variable function
    - $\dfrac{\mathrm{d}}{\mathrm{d}t} f_1(\vec{r}_1(t)) = \nabla f_1 \cdot \vec{r}_1'(t) = (2(t-4), 2t^3) \cdot (1, 3t^2) = 2t - 8 + 6t^5$
    - Let $f_1'(t) = 0 \implies t(1 + 3t^4) = 4 \implies t = 1$ or the point $(1, 1)$, which is in our set
    - $f(1, 1) = 10$
    - $f_1''(t) = 2 + 30t^4 = 32 > 0$ so this is a minimum
  - Boundary 2: $y = 4x, 0 \leq x \leq 2$
    - Parameterize the boundary $\begin{cases} x = t \\ y = 4t \end{cases}$
    - $f_2(t) = (t-4)^2 + (4t)^2 = 17t - 8t + 16 \implies f_2'(t) = 34t - 8 = 0 \implies t = \dfrac{4}{17}$
    - Critical point is at $\left(\dfrac{4}{17}, \dfrac{16}{17}\right)$
  - Test for whether these are minimums or maximums with a second derivative test
  - Check endpoints $f(0, 0) = 16, f(2, 8) = 68$
- Example: $f(x, y) = xy^2 - x$ on $\{ (x, y) \mid x^2 + y^2 \leq 3 \}$
  - $\nabla f = (y^2 - 1)\hat{i} + 2xy\hat{j} = \vec{0} \implies \begin{cases} y^2 - 1 = 0 \\ 2xy = 0 \end{cases} \implies \begin{cases} y = \pm 1 \\ x = 0 \end{cases}$
  - Using second derivative test, both are saddle points
  - Check boundary $y^2 = 3 - x^2 \implies f_1(x) = x(3 - x^2) - x = 2x - x^3 \implies f_1'(x) = 2 - 3x^2 \implies x = \pm\sqrt{\dfrac{3}{2}}, y = \pm\sqrt{\dfrac{7}{3}}$
  - There are 4 critical points on the boundary; using the second derivative test we can determine which one is max or min
  - Even though the circle has no end points, when we expressed it as $y^2 = 3 - x^2$ we introduced constraints of $-\sqrt{3} \leq x \leq \sqrt{3}$, so we must treat those as end points of the boundary
  - Another approach is to parameterize the boundary curve as $\vec{r}(t) = \sqrt{3}\cos t\hat{i} + \sqrt{3}\sin t\hat{j}, 0 \leq t \leq 2\pi$, and then taking the derivative of $f(\vec{r}(t))$, which removes the need for end points