Lecture 32, Apr 1, 2022

Multivariable Optimization

- Definition: f has a local maximum at $\vec{x}_0$ iff $f(\vec{x}_0) \geq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$; f has a local minimum iff $f(\vec{x}_0) \leq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$
- Theorem: If f has a local extreme value at $\vec{x}_0$ then $\nabla f(\vec{x}_0) = \vec{0}$ or $\nabla f(\vec{x}_0)$ does not exist
  - Proof:
    * Let $g(x) = f(x, y_0)$, a single variable function; this function must have an extreme value at $\vec{x}_0$ if f has an extreme value at $\vec{x}_0$
    * $\frac{\mathrm{d}g}{\mathrm{d}x}(x_0) = 0 = \frac{\partial f}{\partial x}(x_0, y_0)$
  - The tangent plane is horizontal when this happens: Consider the level surface $z = f(x, y)$, let $g(x, y, z) = z - f(x, y) = 0$, then $\nabla g = \hat{k}$ for $f_x = f_y = 0$ so the normal is pointing straight up, which means the tangent plane is horizontal
  - However, this *doesn't* mean that the gradient equalling zero or DNE implies the existence of a local extreme
- Definition: Points where $\nabla f = \vec{0}$ or DNE are *critical points*; where $\nabla f = \vec{0}$ are stationary points; stationary points that are not extrema are *saddle points*
- Example: $f(x, y) = 2x^2 + y^2 - xy - 7y$
  - $\nabla f = (4x - y, 2y - x - 7) = \vec{0} \implies \begin{cases} x = 1 \\ y = 4 \end{cases}$
  - $f(1, 4) = -14$
  - Look in the neighbourhood of $(1, 4)$: in all directions f is a little larger, so this point is a local minimum
- Note the max/min could be an entire curve rather than a single point (e.g. a torus)
- Theorem: Second Derivatives Test: For $f(x, y)$ with continuous second order partials and $\nabla f(\vec{x}_0) = \vec{0}$, set $\begin{cases} A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \\ B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{cases}$ and form the discriminant $D = AC - B^2$, then:
  1. If $D < 0$, then (x_0, y_0) is a saddle point
  2. If $D > 0$ and $A, C > 0$, then (x_0, y_0) is a local minimum
     - Note this is only possible if A and C have the same sign
  3. If $D > 0$ and $A, C < 0$ then (x_0, y_0) is a local maximum
  4. In all other cases, the result is indeterminate
- The second derivative test in 2D is a special case of the more general second derivative test, which looks at the Hessian matrix; if the Hessian is positive definite, then the point is a local minimum; if the Hessian is negative definite, it is a local maximum; if the eigenvalues are mixed positive and negative, then it is a saddle point; in all other cases (zero eigenvalues), it is indeterminate