Lecture 31, Mar 29, 2022

Functions of 3 Variables: Level Surfaces

- Level surface: $f(x, y, z) = c$
- By extension from the 2D case, the gradient is perpendicular to the level surface (tangent plane to the surface)
  - This can be shown in the same way as in the 2D case
  - Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be any curve on the surface
  - $f(x(t), y(t), z(t)) = f(\vec{r}(t)) = c \implies \frac{\mathrm{d}}{\mathrm{d}t} f(\vec{r}(t)) = 0 \implies \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = 0$
- Equation for the tangent plane is given by $(\vec{x} - \vec{x}_0) \cdot \nabla f(\vec{x}_0) = (x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} + (z - z_0)\frac{\partial f}{\partial z} = 0$
- Normal line equation is given by $\vec{r}(q) = \vec{x}_0 + q\nabla f(\vec{x}_0)$
- Example: $xy^2 + 2z^2 = 12$, find normal line at $(1, 2, 2)$
  - $\begin{cases} \frac{\partial f}{\partial x} = y^2 = 4 \\ \frac{\partial f}{\partial y} = 2xy = 4 \\ \frac{\partial f}{\partial z} = 4z = 8 \end{cases} \implies \begin{cases} x = 1 + 4q \\ y = 2 + 4q \\ z = 2 + 8q \end{cases}$
- Example: offset sphere $f = x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$ and ellipsoid $g = x^2 + 3y^2 + 2z^2 = 9$, show that the sphere is tangent to the ellipsoid at $(2, 1, 1)$
  - Show that gradient vectors are parallel
  - $\nabla f = (2x - 8)\hat{i} + (2y - 8)\hat{j} + (2z - 6)\hat{k} \implies \nabla f(2, 1, 1) = (-4, -6, -4)$
  - $\nabla g = 2x\hat{i} + 6y\hat{j} + 4z\hat{k} \implies \nabla g(2, 1, 1) = (4, 6, 4) = -\nabla f(2, 1, 1)$
  - Note we also need to show that the point is on both surfaces