Lecture 30, Mar 28, 2022

The Chain Rule

- Theorem: Multivariable chain rule: $\frac{\mathrm{d}}{\mathrm{d}t}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$
  - Recall that $\vec{r}' = \vec{T}\frac{\mathrm{d}x}{\mathrm{d}t}$ so $\nabla f(\vec{r}) \cdot \vec{r}' = \nabla f \cdot \vec{T}\frac{\mathrm{d}s}{\mathrm{d}t}$
    * $\nabla f \vec{T}$ is the directional derivative in the direction of $\vec{T}$
    * Multiplying the rate of change in a direction by the speed gives the overall rate of change
- Example: Rectangular volume $V = lhd$, let $\frac{\mathrm{d}l}{\mathrm{d}t} = 3\text{m/s}, \frac{\mathrm{d}h}{\mathrm{d}t} = -2\text{m/s}, \frac{\mathrm{d}d}{\mathrm{d}t} = 5\text{m/s}$, what is $\frac{\mathrm{d}V}{\mathrm{d}t}$ at $l = 2\text{m}, h = 3\text{m}, d = 4\text{m}$?
  - Let $\vec{q}(t) = (l, h, d)$
  - $\frac{\mathrm{d}V}{\mathrm{d}t} = \nabla V(\vec{q}(t)) \cdot \vec{q}'(t) = (hd, ld, hl) \cdot (3, -2, 5) = 3hd - 2ld + 5lh = 50\text{m}^3 s$
  - A "position vector" can be something other than just spacial directions
- We can extend this further and make $x(t, s), y(t, s)$ also functions of multiple variables; then $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$ etc (note now we have all partial derivatives)

Revisiting Implicit Differentiation

- How do we get $\frac{\mathrm{d}y}{\mathrm{d}x}$ from an implicit relation $u(x, y) = 0$?
  - Let $x = t, y = y(t)$
- $u = u(t, y(t)) \implies \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$
  - Since $u(t, y(t)) = 0$ we have $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$
  - $x = t \implies \frac{\mathrm{d}x}{\mathrm{d}t} = 1$
  - $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}$
  - $0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$
- Example: $x^4 + 4x^3y + y^4 = 1$
  - Let $u = x^4 + 4x^3y + y^4 - 1 = 0$
  - $\frac{\partial u}{\partial x} = 4x^3 + 12x^2y$
  - $\frac{\partial u}{\partial y} = 4x^3 + 4y^3$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(x + 3y)}{x^3 + y^3}$
- Same result can be obtained by doing implicit differentiation normally, but this method can be easier

Level Curves in 2D

- The gradient is always normal to the level curve (i.e. perpendicular to the tangent of the normal curve)
  - $f(x, y) = c$ is the level curve; let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ describe this curve, then $\vec{t} = \vec{r}'(t)$
  - $f(\vec{r}(t)) = c$ must hold for this to be the level curve
  - $\frac{\mathrm{d}}{\mathrm{d}t}f(\vec{r}(t)) = \nabla f(\vec{r}) \cdot \vec{r}' = \frac{\mathrm{d}}{\mathrm{d}t}c = 0$
  - Therefore $\nabla f \perp \vec{r}'$ or $\nabla f \perp \vec{t}$
- This works for any curve in the form $f(x, y) = c$

- Using this we can obtain the expression for $\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$
  - $\nabla f \cdot \vec{t} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} = 0$
- To get the equation of the tangent line: $(x - x_0, y - y_0) \cdot \nabla f = 0 \implies (x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} = 0$
- Similarly for the normal line: $(x - x_0, y - y_0) \cdot \vec{t} = 0 \implies (x - x_0)\frac{\partial f}{\partial y} - (y - y_0)\frac{\partial f}{\partial x} = 0$
- Example: $x^2 + y^2 = 9$
  - $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y \implies (x - x_0)2x_0 + (y - y_0)2y_0 = 0$
  - Choose e.g. the point $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$
  - Tangent is $\left(x - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} + \left(y - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} = 0 \implies y = \frac{6}{\sqrt{2}} - x$