Lecture 3, Jan 14, 2022

Integration By Parts

- The integral counterpart to the product rule
- $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x)\\ \implies \int (f(x)g'(x) + g(x)f'(x))\,\mathrm{d}x &= \int \frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x)\,\mathrm{d}x\\ \implies \int f(x)g'(x)\,\mathrm{d}x + \int g(x)f'(x)\,\mathrm{d}x &= f(x)g(x)\\ \implies \int f(x)g'(x)\,\mathrm{d}x &= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x\end{aligned}$$
- Let $u = f(x), v = g(x) \implies \mathrm{d}u = f'(x)\,\mathrm{d}x, \mathrm{d}v = g'(x)\,\mathrm{d}x \implies \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$
- Example: $$\begin{aligned}\int x\sin x\,\mathrm{d}x &= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x\\ &= x(-\cos x) - \int 1\cdot(-\cos x)\,\mathrm{d}x\\ &= -x\cos x + \sin x + C\end{aligned}$$
- Integration by parts should simplify the integral; choose $f(x)$ so it becomes simpler when differentiated, and $g(x)$ so that it is easy to integrate
- Example: $$\begin{aligned}&\int \ln x\,\mathrm{d}x\\ \implies\ & x\ln x - \int x\cdot\frac{1}{x}\,\mathrm{d}x\\ \implies\ & x\ln x - x + C\end{aligned}$$
- Integration by parts can be applied multiple times if the resulting integral is still not quite simple
- Sometimes we get back the original integral: $$\begin{aligned}\int e^x\sin x\,\mathrm{d}x &= -e^x\cos x + \int e^x\cos x\,\mathrm{d}x\\ &= -e^x\cos x + e^x\sin x - \int e^x\sin x\,\mathrm{d}x\\ \implies 2\int e^x\sin x\,\mathrm{d}x &= e^x(\sin x - \cos x) + C'\\ \implies \int e^x\sin x\,\mathrm{d}x &= \frac{1}{2}e^x(\sin x - \cos x) + C\end{aligned}$$
  - Remember when subtracting $\int f(x)\,\mathrm{d}x - \int f(x)\,\mathrm{d}x \neq 0$, because there is still an integration constant!
- For definite integrals, $\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,\mathrm{d}x$
  - The first term there is a boundary term
- Example: $$\begin{aligned}\int_0^1 \tan^{-1}x\,\mathrm{d}x &= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x\\ &= \frac{\pi}{4} - \frac{1}{2}\int_{t=0}^{t=1}\frac{1}{u}\,\mathrm{d}u\\ &= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1\\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2\end{aligned}$$
- Using integration by parts, we can prove reduction formulas:
  - $\int \sin^n x\,\mathrm{d}x = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,\mathrm{d}x,\ n \geq 2$

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\begin{aligned}
\int \sin^n x \,\mathrm{d}x &= \sin^{n-1} x \sin x \,\mathrm{d}x \\
&= -\cos x \sin^{n-1} x - \int (n-1) \sin^{n-2} x \cos x \cdot - \cos x \,\mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \,\mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \,\mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \,\mathrm{d}x - (n-1) \int \sin^n x \,\mathrm{d}x \\
\implies n \int \sin^n x \,\mathrm{d}x &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \,\mathrm{d}x \\
\implies \int \sin^n x \,\mathrm{d}x &= -\frac{1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} \int \sin^{n-2} x \,\mathrm{d}x
\end{aligned}
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