Lecture 28, Mar 22, 2022

Continuity of Multivariable Functions

- Definition: A multivariable function f is continuous at $\vec{x}_0$ if $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$
  - Example: $f(x, y) = \frac{x^2 + 3xy}{x^2 - 2y}$ is continuous everywhere except $x^2 - 2y = 0$, i.e. the curve $y = \frac{1}{2}x^2$ is excluded
- Theorem: Continuity of composite functions: If g is continuous at $\vec{x}_0$ and f is continuous at $g(\vec{x}_0)$ then $f(g(\vec{x}))$ is continuous at $\vec{x}_0$
  - As in a vector valued function case, the composition can only go in one direction
- If $f(\vec{x})$ is continuous at $\vec{x}_0$, then it is continuous in both variables; however continuity in both variables does not imply continuity as a whole
  - $f(\vec{x})$ is continuous at $\vec{x}_0 \implies \lim_{x \to x_0} f(x, y_0) = f(x_0, y_0)$ and $\lim_{y \to y_0} f(x_0, y) = f(x_0, y_0)$
  - Example: Consider $f(x, mx) = \frac{m^4 x^6}{x^4 + m^8 x^8}$ from last lecture; the limit exists if approaching from any direction in a straight line so it works for the x and y directions, but any other path yields a different limit
  - Example: $f(x, y) = \begin{cases} \frac{xy^2}{x^3 + y^3} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is continuous along both axes, but along the line $y = 3x \implies f(x, 3x) = \frac{9x^3}{28x^3} = \frac{9}{28}$ so the function is not continuous

Partial Derivatives

- Partial derivatives treat all variables except one as a constant and differentiate with respect to that variable
- Definition: Partial derivative of f with respect to x is $f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$
  - Similarly $f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$
- Interpret as slicing the graph with a plane, and then taking the slope of the tangent line of the curve on that plane
- Since partial derivatives are also functions of multiple variables, higher order partial derivatives can be taken, with respect to the same variable or different variables
  - We can mix partial derivatives, e.g. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
  - Note in the notation $\frac{\partial^2 f}{\partial y \partial x}$, the derivative is first taken with respect to x and then y, i.e. denominator is read right-to-left; in the notation f_{xy}, the subscript is read left-to-right, i.e. derivative is taken with respect to x and then y
- Clairaut's Theorem (symmetry of second partial derivatives): $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ on every open set for which $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are continuous
  - This applies to 3 variables as well for $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$, etc

Partial Differential Equations

- When derivatives of more than one variable occur in a relation, it is a partial differential equation
- Example: Laplace's Equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (fluid flow)

- Example: 1D wave equation: $\dfrac{\partial^2 f}{\partial t^2} = a^2 \dfrac{\partial^2 f}{\partial x^2}$