Lecture 27, Mar 21, 2022

Multivariable Functions

- The domain of a multivariable function is an entire region
- One way to visualize a multivariable function is to use a *level map* or *contour plot* (i.e. sections)
  - Cut through the surface with a plane parallel to the xy plane, i.e. set $f(x, y) = c$
  - This gives us a curve and we can plot several of these contours for different c on the same plot
  - This is referred to as a *collection of level curves*
- Example: $f(x, y) = xy$
  - Set $f = c \implies xy = c \implies y = \frac{c}{x}$
  - The level curves are a collection of $\frac{c}{x}$; for positive c the curve is in the first and third quadrant, for negative c the curve is in the second and fourth quadrant
  - The curve looks like a saddle increasing along positive xy and decreasing along negative xy
- For a function of 3 variables, we can extend this idea to make *level surfaces*, so we can get and idea of what the function looks like without a fourth dimension

Limits of Multivariable Functions

- Notation: $f(x, y, z) = f(\vec{x})$
- There are an infinite number of ways to approach a point in multiple dimensions
- Definition: $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < \|\vec{x} - \vec{x}_0\| < \delta \implies |f(\vec{x}) - L| < \varepsilon$
  - δ defines a circle, sphere, etc around the point $\vec{x}_0$
  - In the single variable case we only had left and right hand limits; with multiple variables there are an infinite number of ways
- Example: $\lim_{x \to \vec{0}} \frac{x^2 y + y^2}{x + y^2}$
  - Try approaching the origin from the positive y axis:
    - $x = 0 \implies f(0, y) = \frac{y^2}{y^2} = 1 \implies \lim_{y \to 0^+} f(0, y) = 1$
  - Approaching from the x axis:
    - $y = 0 \implies f(x, 0) = \frac{0}{x} = 0 \implies \lim_{x \to 0^+} f(x, 0) = 0$
  - This limit does not exist because we get different values for it when we approach from different directions
- Example: $\lim_{\vec{x} \to \vec{0}} \frac{x^2 y^4}{x^4 + y^8}$
  - Consider the path $y = mx$:
    - $f(x, mx) = \frac{m^4 x^6}{x^4 + m^8 x^8} = \frac{m^4 x^2}{1 + m^8 x^4} \implies \lim_{x \to 0} f(x, mx) = 0$
  - Consider a parabolic path $x = y^2$:
    - $f(y^2, y) = \frac{y^4 y^4}{y^8 + y^8} = \frac{y^8}{2y^8} = \frac{1}{2} \implies \lim_{y \to 0} f(y^2, y) = \frac{1}{2}$
  - Again the limit does not exist
- To prove that the limit actually does exist we need an epsilon delta proof
- Example: $\lim_{\vec{x} \to \vec{0}} \frac{2xy^2}{x^2 + y^2}$
  - Impose $\varepsilon > 0$, require $|f - L| = \left|\frac{2xy^2}{x^2 + y^2} - 0\right| = \frac{2y^2|x|}{x^2 + y^2} < \varepsilon$, when $0 < \|\vec{x} - \vec{x}_0\| = \sqrt{x^2 + y^2} < \delta$

- $$y^2 \le x^2 + y^2$$
  $$\implies \frac{y^2}{y^2} \ge \frac{y^2}{x^2 + y^2}$$
  $$\implies \frac{2y^2|x|}{x^2 + y^2} \le \frac{2y^2|x|}{y^2} = 2|x|$$
  $$\implies 2|x| = 2\sqrt{x^2} \le 2\sqrt{x^2 + y^2} < 2\delta$$
  * Therefore choose $\delta = \frac{\varepsilon}{2} \implies \left|\frac{2y^2x}{x^2 + y^2}\right| < \varepsilon$
- Therefore $\lim_{\vec{x}\to\vec{0}} \frac{2xy^2}{x^2 + y^2} = 0$