Lecture 25, Mar 15, 2022

Motion in Space: Velocity and Acceleration

- Interpret $\vec{r}(t)$ as the location, then $\vec{r}'(t) = \vec{v}(t)$ is the velocity and $\vec{r}''(t) = \vec{a}(t)$ is the acceleration and $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\vec{r}'(t)\|$ is the speed
- For circular motion around the origin $\vec{r}(t) = a\cos(\theta(t))\hat{\imath} + a\sin(\theta(t))\hat{\jmath}$
  - θ' is the rate of change of the angle, so $\theta' > 0$ is a counterclockwise movement
  - Define θ' as the *angular velocity* in radians per second; $|\theta'|$ is the *angular speed*
    - Note θ' is essentially a scalar property but with a sign
  - If $\theta' = \omega$ then

$$\begin{aligned} & \theta = \omega t \\ \implies & \vec{r} = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath} \\ \implies & \vec{v} = -r\omega\sin(\omega t)\hat{\imath} + r\omega\cos(\omega t)\hat{\jmath} \\ \implies & \vec{a} = -r\omega^2\cos(\omega t)\hat{\imath} - r\omega^2\sin(\omega t)\hat{\jmath} = -\omega^2\vec{r} \end{aligned}$$

Vector Mechanics

- Newton's second law: $\vec{F}(t) = m\vec{a}(t)$
- Momentum: $\vec{p}(t) = m\vec{r}'(t)$ and $\vec{p}'(t) = \vec{F}(t)$
- Conservation of momentum: $\vec{p}' = \vec{F} = 0 \implies \vec{p}$ is constant
- Angular momentum: $\vec{L} \equiv \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ where $\vec{r}$ is the radius vector
  - $\left\|\vec{L}\right\| = m\|\vec{r}\|\|\vec{v}\|\sin(\theta(t))$ where θ is the angle between $\vec{v}$ and $\vec{r}$
  - We can break up $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$ (directions perpendicular and parallel to $\vec{r}$)
  - $\vec{L} = m\vec{r} \times \vec{v} = m\vec{r} \times (\vec{v}_\perp + \vec{v}_\parallel) = m\vec{r} \times \vec{v}_\perp$ since $\vec{r} \times \vec{v}_\parallel = 0$
  - Angular momentum only cares about the component of $\vec{v}$ perpendicular to $\vec{r}$
- Example: $\vec{r}(t) = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath}$
  - $\vec{L} = m\vec{r} \times \vec{v} = m(0, 0, r^2\omega\cos^2(\omega t) + r^2\omega\sin^2(\omega t)) = (0, 0, mr^2\omega)$
- Example: Uniform motion in a straight line
  - $\vec{r} = \vec{r}_0 + t\vec{v}$
  - $\vec{L} = m\vec{r} \times \vec{v} = m(\vec{r}_0 + t\vec{v}) \times \vec{v} = m\vec{r}_0 \times \vec{v}$ which is a constant assuming $\vec{v}$ is constant
  - Uniform motion in a straight line has a constant angular momentum
- Torque: $\vec{L}' = m\vec{r} \times \vec{r}'' + m\vec{r}' \times \vec{r}' = \vec{r} \times \vec{F} \equiv \vec{\tau}$
- Central force: Definition: $\vec{F}$ is a *central* or *radial force* if $\vec{F}(t)$ is always parallel to $\vec{r}$
  - Example: gravity, electric field associated with a point charge
  - $\vec{r} \times \vec{F} = 0 \implies \tau = 0 \implies \vec{L}$ is constant
  - Angular momentum is conserved when only a central force is present

Acceleration

- Break apart acceleration $\vec{a} = \vec{a}_n + \vec{a}_t$ (normal and tangential directions)
- $\vec{T} = \dfrac{\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}}{\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\|} = \dfrac{\vec{v}}{\frac{\mathrm{d}s}{\mathrm{d}t}} \implies \vec{v} = \dfrac{\mathrm{d}s}{\mathrm{d}t}\vec{T}$
  - Differentiating, we get $\vec{v}' = \vec{a} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{T} + \dfrac{\mathrm{d}s}{\mathrm{d}t}\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}$
  - Note $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|\vec{N}$ and $\kappa = \dfrac{\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|}{\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|}$
  - $\vec{a} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{T} + \kappa\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right)^2\vec{N}$
    - The first term is $\vec{a}_t$, the tangential component, the second is $\vec{a}_n$, the normal component
- $\|\vec{a}_t\| = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ which is the rate of change of speed in the direction of motion

  - Example: uniform circular motion with $\frac{\mathrm{d}s}{\mathrm{d}t}$ equal to a constant so $\|\vec{a}_t\| = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$ so there is no tangential acceleration

- $\|\vec{a}_n\| = \kappa \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$ directed perpendicular to the tangent, in the direction of curvature
  - Alternatively $\|\vec{a}_n\| = \frac{\|\vec{v}\|^2}{\rho}$ where $\rho = \frac{1}{\kappa}$ is the radius of curvature