Lecture 24, Mar 14, 2022

Curvature in 2D

- Definition: Curvature is defined as $\kappa \equiv \left|\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|$, where in 2D ϕ is the angle that the tangent line makes with the x axis and s is arc length
  - Let $y = y(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = y' = \tan\phi \implies \phi = \tan^{-1} y' \implies \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{y''}{1 + (y')^2}$
  - Note $\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (y'(x))^2}$
  - $\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\sqrt{1 + (y')^2}$
  - $\frac{\mathrm{d}\phi}{\mathrm{d}s}\sqrt{1 + (y')^2} = \frac{y''}{1 + (y')^2} \implies \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}}$
  - Therefore $\kappa = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$ (all derivatives are with respect to x)
  - For a parametric curve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'}{x'} \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{x'y'' - y'x''}{(x')^3} \implies \kappa = \frac{\left|\frac{x'y''-y'x''}{(x')^3}\right|}{\left(1 + \left(\frac{y'}{x'}\right)^2\right)^{\frac{3}{2}}}$
    $$= \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$
- Example: circle $\vec{r} = r\cos(t)\hat{\imath} + r\sin(t)\hat{\jmath}$
  - $x(t) = r\cos t \implies x'(t) = -r\sin t \implies r'' = -r\cos t$
  - $y(t) = r\sin t \implies y' = r\cos t \implies y'' = -r\sin t$
  - $\kappa = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}}$
    $$= \frac{|r^2\sin^2 t + r^2\cos^2 t|}{(r^2\sin^2 t + r^2\cos^2 t)^{\frac{3}{2}}} = \frac{r^2}{r^3} = \frac{1}{r}$$
  - We can also get this intuitively by noting that $\frac{\Delta\phi}{\Delta s} = \frac{2\pi}{2\pi r} = \frac{1}{r}$
- This leads to the definition for the *radius of curvature*: $\rho = \frac{1}{\kappa}$
  - The radius of curvature is the radius of the tangent circle to the curve at any given point
- Note: Consider the unit tangent vector $\vec{T} = \cos(\phi)\hat{\imath} + \sin(\phi)\hat{\jmath}$
  - $\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} = -\sin(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}s}\hat{\imath} + \cos(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}s}\hat{\jmath}$
  - $\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}s}\right| = \kappa$
  - This gives an alternative interpretation of curvature as the rate of change of the unit tangent vector with respect to arc length

Curvature in 3D

- Definition: $\kappa \equiv \left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\|$ in 3D, where $\vec{T}$ is the unit tangent vector to the curve and s is the arc length
- Consider $C : \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

- $\kappa = \left\| \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}s} \right\| = \frac{\left\| \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} \right\|}{\left| \frac{\mathrm{d}s}{\mathrm{d}t} \right|} = \frac{\left\| \vec{T}' \right\|}{\|\vec{r}'\|}$
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{r}'(t) = 3\cos(t)\hat{\imath} - 3\sin(t)\hat{\jmath} + 4\hat{k}$
  - $\|\vec{r}'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$
  - $\vec{T} = \frac{\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}}{\left\| \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \right\|} = \frac{3}{5}\cos(t)\hat{\imath} - \frac{3}{5}\sin(t)\hat{\jmath} + \frac{4}{5}\hat{k}$
  - $\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} = -\frac{3}{5}\sin(t)\hat{\imath} - \frac{3}{5}\cos(t)\hat{\jmath} + 0\hat{k} \implies \left\| \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} \right\| = \frac{3}{5}$
  - $\kappa = \frac{\frac{3}{5}}{5} = \frac{3}{25}$ or $\rho = \frac{25}{3}$
- Alternatively: $\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

The Normal and Binormal Vectors

- Definition: The *principal unit normal* $\vec{N}(t) \equiv \frac{\vec{T}'(t)}{\left\| \vec{T}'(t) \right\|}$, i.e. a unit vector in the direction of $\vec{T}'$
  - $\vec{T}$ is tangent to the curve and $\vec{N}$ is perpendicular to this tangent
  - $\vec{N}$ points in the direction that the curve is changing
- Definition: The *osculating plane* is the plane defined by $\vec{N}$ and $\vec{T}$
  - The osculating plane is the plane that comes closest to containing the curve at a point
- Definition: The *binormal vector* is $\vec{B} = \vec{T} \times \vec{N}$
- Example: for a straight line $\vec{T}'(t) = 0$ since the tangent vector doesn't change; this means $\vec{N}$ does not exist and we can't define an osculating plane
  - This can be interpreted as a straight line is contained in an infinite number of planes
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{T}'(t) = -\frac{3}{5}\sin(t)\hat{\imath} - \frac{3}{5}\cos(t)\hat{\jmath} \implies \left\| \vec{T}'(t) \right\| = \frac{3}{5}$
  - $\vec{N}(t) = -\sin(t)\hat{\imath} - \cos(t)\hat{\jmath}$
  - $\vec{B} = \left(\frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5} \right)$
  - A point on the plane is $(3\sin t, 3\cos t, 4t)$
  - The equation of the plane is $\left(\frac{4}{5}\cos t \right)(x - 3\sin t) - \left(\frac{4}{5}\sin t \right)(y - 3\cos t) - \frac{3}{5}(z - 4t) = 0 \implies 4\cos(t)x - 4\sin(t)y - 4z = -12t$
- In general the magnitude of the binormal vector is always 1, because $\left\| \vec{B} \right\| = \left\| \vec{T} \right\| \left\| \vec{N} \right\| \sin\theta = 1$
  - $\vec{T}, \vec{N}, \vec{B}$ form a set of mutually perpendicular unit vectors
  - We can use this as a new coordinate system, useful in some physical situations e.g. satellites