Lecture 23, Mar 11, 2022

Vector Derivatives and Integrals

- The derivative of a vector function is defined as $\vec{f}'(t) \equiv \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$
- Derivative can be taken componentwise: $\vec{f}'(t) = f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k}$
  - Proof: $$\begin{aligned}\vec{f}'(t) &= \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f_1(t+h) - f_1(t)}{h}\hat{\imath} + \frac{f_2(t+h) - f_2(t)}{h}\hat{\jmath} + \frac{f_3(t+h) - f_3(t)}{h}\hat{k}\right] \\ &= \lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h}\hat{\imath} + \lim_{h \to 0} \frac{f_2(t+h) - f_2(t)}{h}\hat{\jmath} + \lim_{h \to 0} \frac{f_3(t+h) - f_3(t)}{h}\hat{k} \\ &= f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k}\end{aligned}$$
- Integrals can also be defined componentwise as $\int_a^b \vec{f}(t)\,\mathrm{d}t = \left(\int_a^b f_1(t)\,\mathrm{d}t\right)\hat{\imath} + \left(\int_a^b f_2(t)\,\mathrm{d}t\right)\hat{\jmath} + \left(\int_a^b f_3(t)\,\mathrm{d}t\right)\hat{k}$
- All ordinary derivative and integral properties apply:
  - $\int_a^b \vec{c} \cdot \vec{f}(t)\,\mathrm{d}t = \vec{c} \cdot \int_a^b \vec{f}(t)\,\mathrm{d}t$
  - $\left\|\int_a^b \vec{f}(t)\,\mathrm{d}t\right\| \leq \int_a^b \left\|\vec{f}(t)\right\|\mathrm{d}t$

Differentiation Formulas

- Define a composition function $(\vec{f} \circ u)(t) = \vec{f}(u(t))$
  - Note this composition can't go the other way around, because $\vec{f}$ takes in a scalar and u takes in a vector, so $u(\vec{f}(t))$ makes no sense
- Differentiation rules:
  1. $(\vec{f} + \vec{g})'(t) = \vec{f}'(t) + \vec{g}'(t)$
  2. $(\alpha\vec{f})'(t) = \alpha\vec{f}'(t)$
  3. $(u\vec{f})'(t) = u(t)\vec{f}'(t) + u'(t)\vec{f}(t)$
  4. $(\vec{f} \cdot \vec{g})'(t) = \vec{f}(t) \cdot \vec{g}'(t) + \vec{f}'(t) \cdot \vec{g}(t)$
  5. $(\vec{f} \times \vec{g})'(t) = \vec{f}(t) \times \vec{g}'(t) + \vec{f}'(t) \times \vec{g}(t)$
     - Note that for this one, order matters since cross product is non-commutative
  6. $(\vec{f} \circ u)'(t) = u'(t)\vec{f}(u(t))$
- Example: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
  - Define $r \equiv \|\vec{r}\| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{x^2 + y^2 + z^2} \implies \vec{r} \cdot \vec{r} = r^2$
  - $\vec{r} \cdot \vec{r} = r^2 \implies \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \vec{r} + \vec{r} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = 2r\frac{\mathrm{d}r}{\mathrm{d}t} \implies \vec{r} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = r\frac{\mathrm{d}r}{\mathrm{d}t}$
- Example: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r}}{r}$
  - This is a unit vector in the direction of $\vec{r}$; even though the magnitude is constant, the derivative can be nonzero since the direction can change

- $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r}}{r} = \frac{1}{r}\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\vec{r}$
  $= \frac{1}{r^3}\left(r^2\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - r\frac{\mathrm{d}r}{\mathrm{d}t}\vec{r}\right)$
  $= \frac{1}{r^3}\left((\vec{r}\cdot\vec{r})\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \left(\vec{r}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)\vec{r}\right)$
  $= \frac{1}{r^3}\left(\left(\vec{r}\times\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)\times\vec{r}\right)$
  - Note we used the relationship $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$

Curves

- $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$
- The derivative $\vec{r}'(t)$ is interpreted geometrically as a vector pointing in the tangent direction of the curve
- Definition: Let C be parameterized by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ and be differentiable; then $\vec{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$ (if not $\vec{0}$) is tangent to C at $(x(t), y(t), z(t))$ and $\vec{r}'(t)$ points in the direction of increasing t
- Example: Find tangent to $\vec{r}(t) = (1+2t)\hat{\imath} + t^3\hat{\jmath} + \frac{t}{2}\hat{k}$ at $(9, 64, 2)$
  - First find the t value: $\vec{r}(4) = (9, 64, 2)$
  - $\vec{r}'(t) = 2\hat{\imath} + 3t^3\hat{\jmath} + \frac{1}{2}\hat{k} \implies \vec{r}'(4) = 2\hat{\imath} + 48\hat{\jmath} + \frac{1}{2}\hat{k}$
  - The tangent line is $\vec{R}(q) = 9\hat{\imath} + 64\hat{\jmath} + 2\hat{k} + q\left(2\hat{\imath} + 48\hat{\jmath} + \frac{1}{2}\hat{k}\right)$
- Define the *unit tangent vector* as $\vec{T}(t) \equiv \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$
  - Note $\vec{T}(t)\cdot\vec{T}(t) = 1$ since $\vec{T}$ is a unit vector
  - Differentiating this leads to $\vec{T}'(t)\cdot\vec{T}(t) = 0$
  - $\vec{T}'(t)$ is always in the perpendicular direction to $\vec{T}$; this is because $\vec{T}$ has a constant magnitude so the derivative can only represent a change in direction, which is always perpendicular
  - $\vec{T}'(t)$ is telling you the direction that the curve is curving, similar to how the second derivative tells you whether the function is concave up or down

Arc Length

- Extended to 3D, arc length is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,\mathrm{d}t = \int_a^b \|\vec{r}'(t)\|\,\mathrm{d}t$
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{r}'(t) = 3\cos(t)\hat{\imath} - 3\sin(t)\hat{\jmath} + 4\hat{k}$
  - $\|\vec{r}'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$
  - $\int_0^2 \pi\|\vec{r}'(t)\|\,\mathrm{d}t = 10\pi$
- Sometimes a curve is parameterized with respect to arc length instead of t
- Example: $\vec{r}(t) = t^2\hat{\imath} + t^2\hat{\jmath} - t^2\hat{k}$ from $(0, 0, 0)$
  - $s = \int_0^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau$
    $= \int_0^t \sqrt{4\tau^2 + 4\tau^2 + 4\tau^2}\,\mathrm{d}\tau$
    $= \int_0^3 2\sqrt{3}\tau\,\mathrm{d}\tau$
    $= \sqrt{3}t^2$

- $s = \sqrt{3}t^2 \implies t^2 = \frac{s}{\sqrt{3}} \implies \vec{r}(s) = \frac{s}{\sqrt{3}}\hat{i} + \frac{s}{\sqrt{3}}\hat{j} - \frac{s}{\sqrt{3}}\hat{k}$