Lecture 22, Mar 8, 2022

Surfaces: Cylinders and Quadric Surfaces

- Quadric surfaces are in the form $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$
  - $Gx + Hy + Iz$ offsets the shape in the axial directions
  - $Dxy + Exz + Fyz$ terms rotate the shape
- Cylinders are 2D surfaces with no z defined
- Properties of quadric surfaces:
  1. Domain/range
     - Typically domain involves x and y, range involves z, but not always
  2. Intercepts with coordinate axes (all 3)
  3. Traces – intersections with coordinate planes
  4. Sections – intersections with other planes (typically a plane parallel to the coordinate planes)
  5. Centre
  6. Symmetry
  7. Bounded/unboundedness
- Example: Hyperboloid of two sheets: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1 \implies z = \pm c\sqrt{1 + \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$
  1. Domain: $x, y \in (-\infty, \infty)$; range: $z \geq c$ or $z \leq -c$ (since the number in the root is always greater than 1)
  2. Intercepts: z intercept at $z = \pm c$ (when $x = y = 0$) (we can't have $z = 0$ so it can't have other intercepts)
  3. Traces:
     - xy plane ($z = 0$): nothing
     - xz plane ($y = 0$): hyperbola $z = \pm c\sqrt{1 + \dfrac{x^2}{a^2}}$
     - yz plane ($x = 0$): hyperbola $z = \pm c\sqrt{1 + \dfrac{y^2}{b^2}}$
  4. Sections: consider $z = z_0$ where $|z_0| > c$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z_0^2}{c^2} - 1$ where $\dfrac{z_0^2}{c^2} - 1 > 0$, which is an ellipse
  5. Centre: origin because there are no offset terms
  6. Symmetry: $x \to -x$, $y \to -y$, $z \to -z$ all give the same surface so symmetry about each axis
  7. Unbounded in all directions

Projections

- For 2 intersecting surfaces, we will have a curve of intersection $C : (x, y, z)$ such that $z = f(x, y) = g(x, y)$
  - The relation $f(x, y) = g(x, y)$ is a vertical cylinder since there are no restrictions on z
- Fix $z = 0$ and have $f(x, y) = g(x, y) = 0$; this is a *projection* (like a shadow of the curve of intersection)
- Example: cone $x^2 + y^2 = 2z^2 \implies z = \pm\sqrt{\dfrac{x^2 + y^2}{2}}$ and a plane $y + 4z = 5 \implies z = \dfrac{5 - y}{4}$
  - Set
    $$\begin{aligned} &\sqrt{\frac{x^2 + y^2}{2}} = \frac{5 - y}{4} \\ \implies &(5 - y)^2 = 8(x^2 + y^2) \\ \implies &25 - 10y + y^2 = 8x^2 + 8y^2 \\ \implies &8x^2 + 7y^2 + 10y - 25 = 0 \\ \implies &\frac{x^2}{\frac{25}{7}} + \frac{\left(y + \frac{5}{7}\right)^2}{\frac{200}{49}} = 1 \end{aligned}$$
  - This is an offset ellipse

Vector Functions and Limits

- A vector valued function or vector function is $\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k} = (f_1(t), f_2(t), f_3(t))$
  - Like a parametric representation
- Definition: $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$ if $\lim_{t\to t_0} \left\|\vec{f}(t) - \vec{L}\right\| = 0$
  - The norm turns the vector limit into an ordinary limit
- $\lim_{t\to t_0} \vec{f}(t) = \vec{L} \implies \lim_{t\to t_0} \left\|\vec{f}(t)\right\| = \left\|\vec{L}\right\|$ but the reverse is not true
- Limit rules: Given $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$, $\lim_{t\to t_0} \vec{g}(t) = \vec{M}$, $\lim_{t\to t_0} u(t) = R$:
  1. $\lim_{t\to t_0} \vec{f}(t) + \vec{g}(t) = \vec{L} + \vec{M}$
  2. $\lim_{t\to t_0} \alpha\vec{f}(t) = \alpha\vec{L}$
  3. $\lim_{t\to t_0} u(t)\vec{f}(t) = R\vec{L}$
  4. $\lim_{t\to t_0} \vec{f}(t) \cdot \vec{g}(t) = \vec{L} \cdot \vec{M}$
  5. $\lim_{t\to t_0} \vec{f}(t) \times \vec{g}(t) = \vec{L} \times \vec{M}$
- $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$ iff $\lim_{t\to t_0} f_1(t) = L_1$, $\lim_{t\to t_0} f_2(t) = L_2$, $\lim_{t\to t_0} f_3(t) = L_3$
- $\vec{f}(t)$ is continuous at t_0 if $\lim_{t\to t_0} \vec{f}(t) = \vec{f}(t_0)$
- Likewise the continuity of $\vec{f}$depends on the continuity of its component