Lecture 20, Mar 4, 2022

The Binomial Series

- Binomial theorem: $(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$ where $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} = \frac{k!}{(k-n)!n!}$ and $\binom{k}{0} = 1$
- Consider $(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$ which is a power series
- With the binomial theorem, we assumed that k is a positive integer; now we show that this works for all values of k but the series becomes infinite
- Using the Maclaurin series for $(1+x)^k$:
  - Derivatives:
    * $f(x) = (1+x)^k \implies f(0) = 1$
    * $f'(x) = k(1+x)^{k-1} \implies f'(0) = k$
    * $f''(x) = k(k-1)(1+x)^{k-2} \implies f''(0) = k(k-1)$
    * $f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$
  - $f(x) = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$
  - Notice that if k is a positive integer, then at $n = k+1$ we get a zero term in the derivative, which truncates the series
- Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k-n}{n+1}x\right| \underset{n\to\infty}{\to} |x|$
  - The binomial series converges for $|x| < 1$
  - For the endpoints:
    * $k \leq -1 \implies I = (-1, 1)$
    * $-1 < k < 0 \implies I = (-1, 1]$
    * $k \geq 0 \implies I = [-1, 1]$
- Example: $f(x) = \frac{1}{\sqrt{2+x}} = (2+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}\left(1 + \frac{x}{2}\right)^{-\frac{1}{2}} \implies k = -\frac{1}{2}$
  - $f(x) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{x}{2}\right)^n = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\frac{x}{2} + \frac{3}{4}\frac{\left(\frac{x}{2}\right)^2}{2!} + \cdots\right)$
  - Radius of convergence: $\left|\frac{x}{2}\right| < 1 \implies |x| < 2, R = 2$
  - Since $-1 < k = -\frac{1}{2} < 0$ the interval of convergence is $(-2, 2]$
- Linearization: If we take the first degree Taylor polynomial we get $(1+x)^k \approx 1 + kx$

Fourier Series

- Fourier series allow us to represent any periodic function as an infinite sum of sines and cosines
- Definition: A function is periodic with period T if $f(t+T) = f(t)$ for all t
  - The smallest T for which this holds is called the *fundamental period*
  - e.g. $\cos(\pi t)$ has a fundamental period of 2; $\sin(2\pi t)$ has a fundamental period of 1; $\cos(\pi t) + \sin(2\pi t)$ has a fundamental period of 2, the larger of the 2 periods
- Theorem: Let $f(t)$ be a piecewise continuous and piecewise differentiable periodic function with fundamental period T, then $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$ where $\omega = \frac{2\pi}{T}$ (known as the *Fourier* series of f)
  - Note: Piecewise differentiability means we only have a finite number of places where the function is not differentiable
  - a_0 is divided by 2 to allow us to write a single definition for the a_ns

- a_n, b_n are known as the Fourier coefficients; finding a Fourier series is all about finding these coefficients
- Note $\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & n \neq 0 \\ T & n = 0 \end{cases}$ as the positive and negative parts exactly cancel each other out
- Similarly $\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(n\omega t)\,\mathrm{d}t = 0$
- $\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(m\omega t)\sin(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \end{cases}$
- $\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(m\omega t)\sin(n\omega t)\,\mathrm{d}t = 0$
    - See section 7.2
- To find the Fourier coefficients, we multiply the series through by sine or cosine and then integrate, so that all terms except for the one we want go to zero
    - $\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(m\omega t)\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} a_m \cos(m\omega t)\cos(m\omega t)\,\mathrm{d}t + \sum \int_{-\frac{T}{2}}^{\frac{T}{2}} a_n \cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t = \frac{T}{2} a_m$
- Fourier coefficients are given by $a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega t)\,\mathrm{d}t, b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega t)\,\mathrm{d}t$
    - $n = 0, 1, \cdots$ for a_n but starts at 1 for b_n
        * For $n = 0$, b_0 is just 0 so we skip it
        * Notice that for $n = 0$, $a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t$ which is just the average value of f over one period (times 2)
- Example: Triangle function with height of π and period of 2π, with peak at the origin
    - $T = 2\pi, \omega = \frac{2\pi}{T} = 1$
    - The function can be represented by $f(t) = \pi - |t|, t \in [-\pi, \pi]$
    - $$\begin{aligned} b_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |t|)\sin(nt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \pi\sin(nt)\,\mathrm{d}t - \int_{-\pi}^{0} (-t)\sin(nt)\,\mathrm{d}t - \int_{0}^{\pi} t\sin(nt)\,\mathrm{d}t\right) \\ &= -\frac{1}{\pi}\left(-\frac{1}{n^2}\left[\sin(nt) - nt\cos(nt)\right]_{-\pi}^{0} + \frac{1}{n^2}\left[\sin(nt) - nt\cos(nt)\right]_{0}^{\pi}\right) \\ &= 0 \end{aligned}$$
        * This makes sense because the triangle function is an even function, but the b_n terms correspond to sines which are odd, so they shouldn't have any contribution
    - $$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega t)\,\mathrm{d}t \\ &= 2\frac{1}{\pi}\int_{0}^{\pi} (\pi - t)\cos(nt)\,\mathrm{d}t \\ &= \begin{cases} \pi & n = 0 \\ 0 & n \neq 0 \text{ and odd} \\ \frac{4}{\pi n^2} & n \neq 0 \text{ and even} \end{cases} \end{aligned}$$
    - $f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cos((2k-1)t)$

- Note $f(0) = \pi = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \implies \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

- Example: $f(t) = t^2, t \in [-\pi, \pi]$
  - $T = 2\pi, \omega = \frac{2\pi}{T} = 1$
  - This is an even function so $b_n = 0$
  - $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \, \mathrm{d}t = \frac{2\pi^2}{3}$
  - $a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} t^2 \cos(nt) \, \mathrm{d}t = \begin{cases} -\frac{4}{n^2} & n \text{ odd} \\ \frac{4}{n^2} & n \text{ even} \end{cases}$
  - $f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nt)$