Lecture 19, Mar 1, 2022

Multiplication and Division of Power Series

- Example: $\dfrac{e^x}{1-x}$
  - We can find its Taylor expansion by multiplying two series together: $\left(1 + x + \dfrac{x^2}{2!} + \cdots\right)\left(1 + x + x^2 + \cdots\right)$
- Generally the radius of convergence of a product or ratio is the smaller of the two radii
- Example: $\tan x = \dfrac{\sin x}{\cos x}$
  - $\dfrac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots}$
  - Note here the radius of convergence is $|x| < \dfrac{\pi}{2}$

Applications of Taylor Polynomials

- Error bounds can be found for Taylor approximations:
  1. If it's an alternating series, then like all alternating series the error bound is just the next term
  2. Otherwise, the error bound can be computed using Taylor's theorem $R_n = \displaystyle\int_a^x \frac{(t-a)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t < M\frac{(x-a)^{n+1}}{(n+1)!}$
- Example: Using the Taylor expansion of $\sqrt{x}$ about $a = 1$, evaluate $\sqrt{1.25}$
  - Derivatives:
    * $f(x) = x^{\frac{1}{2}} \implies f(1) = 1$
    * $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} \implies f'(1) = \dfrac{1}{2}$
    * $f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}} \implies f''(1) = -\dfrac{1}{4}$
    * $f'''(x) = \dfrac{3}{8}x^{-\frac{5}{2}} \implies f'''(1) = \dfrac{3}{8}$
    * $f''''(x) = -\dfrac{15}{16}x^{-\frac{7}{2}} \implies f''''(1) = \dfrac{-15}{16}$
  - $\sqrt{x} \approx T_3(x) = 1 + \dfrac{1}{2}(x-1) - \dfrac{1}{4}\dfrac{(x-1)^2}{2!} + \dfrac{3}{8}\dfrac{(x-1)^3}{3!}$
    * Since this is an alternating series: $|R_3(x)| < |a_4| = \dfrac{15}{16}\dfrac{(x-1)^4}{4!}$
  - $\sqrt{1.25} \approx 1.11816 \pm 0.00015$
- Example: Maximum error for the Maclaurin series of $\cos x$ for $|x| < \dfrac{\pi}{4}$ for $n = 3$
  - $\cos x \approx T_3(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$
  - Another alternating series, so the uncertainty is $|R_3(x)| < \dfrac{x^8}{8!} < \dfrac{\left(\frac{\pi}{4}\right)^8}{8!} = 3.6 \times 10^{-6}$
  - Alternatively, using the Taylor remainder formula $|R_3(x)| < 1\left|\dfrac{x^8}{8!}\right|$
- Example: Find $\ln(1.4)$ to within 0.001 with $\ln(1-x)$
  - $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots$
  - When x is negative this is an alternating series so we can use the next term as an error bound
  - $|R_5(x)| = \left|\dfrac{x^6}{6}\right| = \dfrac{0.4^6}{6} = 0.0007$ so we need to take 5 terms