Lecture 18, Feb 28, 2022

Taylor's Theorem

- When is a function actually equal to its Taylor expansion?
  - Define a partial sum for the Taylor series: $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ (the n-th degree Taylor polynomial of f about a)
  - Define the remainder as $R_n(x) = f(x) - T_n(x)$
- Theorem: If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n \to \infty} R_n(x) = 0$ then f is equal to its Taylor series expansion
- Taylor's Theorem: Given $f', f'', \cdots, f^{(n+1)}$ exists and are continuous on an open interval I, and $a \in I$, then $\forall x \in I, f(x) = T_n(x) + R_n(x)$ where $R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t$
  - Proof:
    * Consider: $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$ by FTC
    * Integration by parts: $$\begin{aligned}\int_a^b f'(t)\,\mathrm{d}t &= [f'(t)(t-b)]_a^b - \int_a^b f''(x)(t-b)\,\mathrm{d}t \\ &= f'(a)(b-a) + \int_a^b f''(t)(b-t)\,\mathrm{d}t \\ &= f'(a)(b-a) + \left[-\frac{(b-t)^2}{2}f''(t)\right]_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \\ &= f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2!} + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \\ &= \cdots\end{aligned}$$
    * Applying this n times: $\int_a^b f'(t)\,\mathrm{d}t = \sum_{i=1}^{n} \frac{(b-a)^i}{i!}f^{(i)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t = f(b) - f(a)$
    * Let $x = b \implies f(x) = \sum_{i=0}^{n} \frac{(x-a)^i}{i!}f^{(i)}(a) + R_n(x)$ where $R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t$
- To prove that a function is equal to its Taylor series we need to prove $\lim_{n \to \infty} R_n(x) = \lim_{n \to \infty} \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(x)\,\mathrm{d}t = 0$; the integral form is not always the most convenient to work with
  - If we can bound the derivative: For $|f^{(x+1)}(t)| \leq M$ for $a < t < x$: $$\begin{aligned}|R_n(x)| &\leq \left|\int_a^x \frac{M(x-t)^n}{n!}\,\mathrm{d}t\right| \\ &= \left|M\left[\frac{(x-t)^{n+1}}{(n+1)!}\right]_a^x\right| \\ &= M\frac{|x-a|^{n+1}}{(n+1)!}\end{aligned}$$
  - Or using the MVT, $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ for $c \in (a, x)$
- Example: Prove e^x is equal to the sum of its Taylor series
  - $f^{(n+1)}(t) = e^t$
  - For a Taylor series about 0 the range is $0 < t < x \implies e^t < e^x = M$, or $x < t < 0 \implies e^t < 1 = M$
  - $R_n(x) \leq \frac{e^x x^{n+1}}{(n+1)!} \underset{n \to \infty}{\to} 0$ (by sequence limit 4)
  - Since the remainder goes to 0 the Taylor series converges to e^x for all x

Taylor Series Examples

- Example: Maclaurin series for $\cos x$
  - $f(x) = \cos x \implies f(0) = 1$
  - $f'(x) = -\sin x \implies f'(0) = 0$
  - $f''(x) = -\cos x \implies f''(0) = -1$
  - $f'''(x) = \sin x \implies f'''(x) = 0$
  - $f''''(x) = \cos x$ so the cycle repeats
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ because all the odd terms are zero
  - Use ratio test to determine radius of convergence (all x)
  - Note for all derivatives the magnitude is always ≤ 1: $|R_n(x)| \leq \left|\frac{x^{n+1}}{(n+1)!}\right| \underset{n\to\infty}{\to} 0$ so the Taylor series sum to $\cos x$
- As long as the derivative doesn't tend to infinity, R_n always goes to 0
- Since the coefficients of a Taylor series are unique we can obtain them in other methods; e.g. differentiating the series of $\cos x$ to get $\sin x$ or multiplying by x to get $x \sin x$
- Example: Taylor series for $\cos x$ about $\frac{17\pi}{4}$
  - This series is useful despite the series for $\cos x$ converging for all x due to rate of convergence
  - Derivatives:
    * $f(x) = \cos x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * $f'(x) = -\sin x \implies f\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
    * $f''(x) = -\cos x \implies f\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
    * $f'''(x) = \sin x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * $f''''(x) = \cos x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * There are two negatives and two positives alternating so we need to use 2 sums
  - $\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{17\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{\left(x - \frac{17\pi}{4}\right)^2}{2!} + \cdots$
  - $\cos x = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(-1)^n \frac{\left(x - \frac{17\pi}{4}\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(-1)^{n+1} \frac{\left(x - \frac{17\pi}{4}\right)^{2n+1}}{(2n+1)!}$
- Example: Prove $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ for $x \in (-1, 1]$
  - Derivatives:
    * $f(x) = \ln(1+x)$
    * $f'(x) = \frac{1}{1+x}$
    * $f''(x) = \frac{-1}{(1+x)^2}$
    * $f'''(x) = \frac{2}{(1+x)^3}$
    * $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$
  - We can't bound this derivative in $-1 < x \leq 1$ because as x tends to -1 the derivative shoots off to infinity, so we need to work with the integral form

- $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t$
  $= \frac{1}{n!}\int_0^x (-1)^{n+1}\frac{n!}{(1+t)^{n+1}}(x-t)^n \, \mathrm{d}t$
  $= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t$
- For $0 \le x \le 1$: $|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \le \int_0^x (x-t)^n \, \mathrm{d}t = \frac{x^{n+1}}{n+1} \underset{n\to\infty}{\to} 0$ since $x < 1$
- For $-1 < x < 0$: $|R_n(x)| = \left| \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \right|$
  $= \int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t} \, \mathrm{d}t$
  * Apply MVT: $\int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t} \, \mathrm{d}t = \left(\frac{z-x}{1+z}\right)^n \frac{1}{1+z}(-x)$, where $x < z < 0$ and $-x$ is the interval width $b-a$
  * To show $\frac{z-x}{1+z} < 1$: $|x| < 1$
    $\implies |x| - |z| < 1 - |z|$
    $\implies \frac{|x|-|z|}{1-|z|} < 1$
    $\implies \frac{-x+z}{1+z} < 1$
    $\implies \frac{z-x}{1+z} < 1$
  * $\lim_{n\to\infty} \left(\frac{z-x}{1+z}\right)^n = 0 \implies R_n(x) = \left(\frac{z-x}{1+z}\right)^n \frac{1}{1+z}(-x) \underset{n\to\infty}{\to} 0$