Lecture 17, Feb 18, 2022

Representing Functions as Power Series

- Example: $\dfrac{x}{x-3}$
  - $\dfrac{x}{x-3} = -x \cdot \dfrac{1}{3-x} = -\dfrac{x}{3} \cdot \dfrac{1}{1-\frac{x}{3}} = -\dfrac{x}{3}\left(1 + \dfrac{x}{3} + \left(\dfrac{x}{3}\right)^2 + \cdots\right) = -\sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^{n+1}$
  - Converges for $|x| < 3$
- Theorem: Term-by-Term Differentiation and Integration: For $\sum C_n(x-a)^n$ with radius of convergence $R = R_0 > 0$ then $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ is continuous and differentiable on $(a - R_0, a + R_0)$, and:
  - $f'(x) = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$
    * Note the sum now starts from $n = 1$, because there is a constant term that disappears
    * Alternatively, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} C_n(x-a)^n\right) = \sum_{n=0}^{\infty}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(C_n(x-a)^n\right)$
  - $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty}\dfrac{C_n}{n+1}(x-a)^{n+1}$
    * Alternatively, $\int\left(\sum_{n=0}^{\infty} C_n(x-a)^n\right)\mathrm{d}x = \sum_{n=0}^{\infty}\int\left(C_n(x-a)^n\right)\mathrm{d}x$
  - Both derived series have the same radius of convergence, but the behaviour at end points may change
- This allows us to calculate some otherwise difficult integrals, e.g. $\int_0^{0.1}\dfrac{1}{1+x^4}\,\mathrm{d}x$
- Example: $\dfrac{1}{(1+x)^2}$
  - Note $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{1}{1+x}\right) = \dfrac{1}{(1+x)^2}$
  - $\dfrac{1}{(1+x)^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{1}{1+x}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\sum_{n=0}^{\infty}(-x)^n\right) = \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} = \sum_{n=1}^{\infty}(-1)^n(n+1)x^n$
- Example: $\ln(1-x)$
  - $\ln(1-x) = -\int\dfrac{1}{1-x}\,\mathrm{d}x = -\int\sum_{n=0}^{\infty}x^n\,\mathrm{d}x = C - \sum_{n=0}^{\infty}\dfrac{x^{n+1}}{n+1} = -\sum_{n=0}^{\infty}\dfrac{x^n}{n}$
  - Solve for the constant of integration: Set $x = 0$, $\ln(1-0) = 0 = C$
- Example: $\tan^{-1} x$
  - $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \dfrac{1}{1+x^2}$
  - $\tan^{-1} x = \int\dfrac{1}{1+x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}\,\mathrm{d}x = C + \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1} = C + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$
  - Calculate constant of integration by $x = 0$: $C = \tan^{-1} 0 = 0$
  - $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$
  - Radius of convergence $R = 1$ follows from the original geometric series; we can test the boundaries to see it also converges for $x = \pm 1$
  - This leads to $\tan^{-1} = \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$ (Leibniz formula for π)

Taylor and Maclaurin Series

- Let $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$ for $|x-a| < R$
  - Notice that $f(a) = C_0$
  - $f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots \implies f'(a) = C_1$
  - $f''(x) = 2C_2 + 6C_3(x-a) + 12C_4(x-a)^2 + \cdots \implies f''(a) = 2C_2$
  - Following this pattern we note that $f^{(n)}(a) = n!C_n$
- Theorem: If $f(x)$ has a power series representation about a of $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ for $|x-a| < R$ then the coefficients are given by $C_n = \frac{f^{(n)}(a)}{n!}$
  - $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots$
  - This is known as the *Taylor series* of f about a
  - In the special case where $a = 0 \implies f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$ this is called a Maclaurin series
- Definition: A function is *analytic* at a if it can be represented as a power series about a
  - The function essentially needs to be infinitely differentiable at a
- Example: $f(x) = e^x$ about 0
  - All derivatives at 0 are equal to 1
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  - To determine our ratio of convergence use the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{x}{n+1} \underset{n\to\infty}{\to} 0$ so the series converges for all x