Lecture 16, Feb 15, 2022

The Root and Ratio Tests

- The Root Test: Given $\sum a_k, a_k \geq 0$, if $\lim_{k\to\infty} a_k^{\frac{1}{k}} = \lim_{k\to\infty} \sqrt[k]{a_k} = p$, then
  1. If $p < 1$ then $\sum a_k$ converges
  2. If $p > 1$ then $\sum a_k$ diverges
  3. If $p = 1$ then the test is inconclusive
- Proof of part 1:
  - Given $p < 1$, choose μ such that $p < \mu < 1$
  - $a_k^{\frac{1}{k}} < \mu$ for sufficiently large k since it approaches p
  - $a_k < \mu^k$ for sufficiently large k
  - $\sum \mu^k$ converges because it is a geometric series with $x < 1$, therefore by the comparison test $\sum a_k$ converges
- In effect the root test is a limit comparison test with a geometric series
- Example: $\sum \left(\frac{n^2+1}{2n^2+1}\right)^n$
  - $a_n^{\frac{1}{n}} = \frac{n^2+1}{2n^2+1}$
  - $\lim_{n\to\infty} a_n = \frac{1}{2} < 1$ so the series converges
- The Ratio Test: Given $\sum a_k, a_k > 0$, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$, then
  1. If $\lambda < 1$ then $\sum a_k$ converges
  2. If $\lambda > 1$ then $\sum a_k$ diverges
  3. If $\lambda = 1$ then the test is inconclusive
- Proof of part 1:
  - Given $\lambda < 1$ choose μ such that $\lambda < \mu < 1$
  - Then $\frac{a_{k+1}}{a_k} < \mu$ for some sufficiently large $k > K$
  - $\frac{a_{K+1}}{a_K} < \mu \implies a_{K+1} < \mu a_K \implies a_{K+2} < \mu a_{K+1} < \mu^2 a_K \implies \cdots \implies a_{K+j} < \mu^j a_K$
  - Let $n = K + j \implies a_n < \mu^{n-K} a_K = \frac{a_K}{\mu^K}\mu^n$
  - $\sum a_n < \frac{a_K}{\mu^K} \sum \mu^n$, which is a convergent geometric series since $\mu < 1$, therefore $\sum a_n$ converges as well
- Example: $\sum \frac{k^2}{e^k}$
  - $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \frac{(k+1)^2}{k^2} \cdot \frac{1}{e}$
  - The ratio goes to $\frac{1}{e} < 1$ in the limit so the series is convergent
- Both tests give you absolute convergence

Power Series

- Definition: A *power series* is a series of the form $\sum_{n=0}^{\infty} C_n x^n$ where C_n are the coefficients of the series
- Example: $C_n = 1$ for all n gives the geometric series
- We can generalize the power series to $\sum_{n=0}^{\infty} C_n (x-a)^n$, which is a power series *about a*
  - Note $x^0 = (x-a)^0 = 1$ even when $x = 0$ or $x = a$

- Therefore $x = a \implies \sum_{n=0}^{\infty} C_n(x-a)^n = C_0$ which always converges

- Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \cdot \frac{n^2}{(n+1)^2} \underset{n\to\infty}{\to} |x|$
  - Therefore this series converges absolutely when $|x| < 1$ and diverges $|x| > 1$
  - Special case when $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$ which converges
  - When $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the alternating series test
  - Therefore this series converges for $-1 \le x \le 1$
- Example: $\sum_{n=0}^{\infty} \frac{(1+5^n)x^n}{n!}$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(1+5^{n+1})x^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+5^n)x^n}\right| = \frac{1+5^{n+1}}{1+5^n}\left|\frac{x}{n+1}\right| \underset{n\to\infty}{\to} 0$
  - Therefore this series converges absolutely for all value of x
- Example: $\sum_{n=0}^{\infty} n!x^n$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|x| \underset{n\to\infty}{\to} \infty$
  - This diverges for all value of x, *except* $x = 0$
- Theorem: For a power series of the form $\sum_{n=0}^{\infty} C_n(x-a)^n$ has 3 possibilities:
  1. The series converges only when $x = a$
  2. The series converges for all x
  3. The series converges for $|x - a| < R$
     - R is known as the *radius of convergence*
     - The *interval of convergence* is the interval of x for which the series is convergent; this may or may not include end points
- Note the power series will always converge for $x = a$ no matter what the series is
- Typically the ratio test is used to determine the radius of convergence, but will not work for the end points (which are often considered as special cases)