Lecture 15, Feb 14, 2022

Limit Comparison Test

- Limit Comparison Test: Given $\sum a_k, \sum b_k; a_k > 0, b_k > 0$, then
  1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then both series converge or diverge
  2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then convergence of $\sum b_n$ implies convergence of $\sum a_n$
  3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then divergence of $\sum b_n$ implies divergence of $\sum a_n$
- Case 1 proof:
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = c \implies \left|\frac{a_n}{b_n} - c\right| < \varepsilon$ for $n > N$
  - Choose $\varepsilon = \frac{c}{2} \implies \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \implies \frac{c}{2}b_n \leq a_n \leq \frac{3c}{2}b_n$ for $n \geq N$
  - If $\sum b_n$ converges then so does $\frac{3c}{2}\sum b_n$ so $\sum a_n$ converges by the comparison test
  - If $\sum b_n$ diverges then so does $\frac{c}{2}$ so $\sum a_n$ diverges by the comparison
- Example: $\frac{1}{n^3 - n}$
  - Limit comparison test to $\frac{1}{n^3}$
  - $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3 - n} = 1$ so by LCT, the series converges since $\sum \frac{1}{n^3}$ is convergent

Alternating Series

- Sometimes series contain both positive and negative terms
- An *alternating series* alternates between positive and negative terms
  - Not all series with both positive and negative terms are alternating, e.g. $\frac{\cos n}{n^2}$
- Alternating series usually have a $(-1)^n$ term to make the alternating signs
- Alternating Series Test: Let $\{ a_k \}$ be a sequence of positive numbers; if $a_{k+1} < a_k$ and $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ converges
  - Since the terms alternate between positive and negative and are decreasing we're always bouncing around in a range that's getting smaller
  - Any partial sum must lie between the two previous sums
  - Proof:
    * First look at the even terms: $S_2 = a_1 - a_2 > 0, S_4 = S_2 + (a_3 - a_4) > 0, \cdots, S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2}$
      - By induction, $\{ S_{2n} \}$ is a monotonically increasing sequence
      - Also, $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$; since all the terms after a_1 are positive, $S_{2n} < a_1$ for all n
      - Since $\{ S_{2n} \}$ is monotonic and bounded by the monotonic sequence theorem it converges
      - Let $\lim_{n\to\infty} S_{2n} = L$
    * Now look at the odd terms: $S_{2n+1} = S_{2n} + a_{2n+1}$
      - $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$
      - First limit is L as above, second limit is 0 since we require that the sequence goes to 0, therefore $\lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L$
    * Since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1} = L$, $\lim_{n\to\infty} S_n = L$ so the series converges
  - If $a_n \to 0$ is *not* true, then the series *always* diverges, but the series being monotonically decreasing is *not* a strict requirement for convergence

- Example: Alternating harmonic series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges because absolute value of terms decreases and magnitude goes to 0

Alternating Series Error Bounds

- The properties of an alternating series give us that L will always be between S_n and S_{n+1}: $|L - S_n| \leq a_{n+1}$
- The error in a partial sum is less than the next term in the series

Absolute and Conditional Convergence

- Definition: If $\sum |a_k|$ converges, then $\sum a_k$ is *absolutely convergent*; if $\sum a_k$ converges but not $\sum |a_k|$, then $\sum a_k$ is *conditionally convergent*
- Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ also converges
  - Proof: $-|a_n| \leq a_n \leq |a_n| \implies 0 \leq a_n + |a_n| \leq 2|a_n|$
    * Let $a_n + |a_n| = b_n \implies 0 \leq b_n \leq 2|a_n|$
    * We know $2\sum |a_n|$ converges, therefore $\sum b_n$ converges by the comparison test since $a_n + |a_n| \leq 2|a_n|$
    * Rearranging, $\sum a_n = \sum b_n - \sum |a_n|$
    * Because both $\sum b_n$ and $\sum |a_n|$ is convergent, $\sum a_n$ is convergent
- Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent
- $\sum \frac{(-1)^{k+1}}{k} = \sum \frac{1}{2k-1} - \sum \frac{1}{2k}$, but for conditionally convergent series we have an $\infty - \infty$ situation
  - This means we must be careful when moving the terms around; depending on the rate that both sums approach infinity we can get a different value out of it