Lecture 14, Feb 11, 2022

Infinite Series Theorems

- Limit laws apply for infinite series
  - Sum rule
  - Scalar multiplication
- Theorem: $\sum_{k=0}^{\infty} a_k$ converges iff $\sum_{k=j}^{\infty} a_k$, i.e. convergence only depends on the behaviour of the part extending to infinity
  - Note by definition every term in a_k has to be finite since all terms in a sequence have to be numbers, so we can't have anything infinite in the first j terms
  - $\sum_{k=j}^{\infty} a_k = L - (a_0 + a_1 + \cdots + a_{j-1})$
- Example: Given $\sum_{k=4}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$ converges, then we know $\sum_{k=0}^{\infty} \frac{3^{k-1}}{4^{3k+1}} = \frac{1}{12} + \frac{1}{64} + \frac{3}{47} + \frac{9}{40} + \sum_{k=4}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$ also converges
- Theorem: If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$
  - Contrapositive: If $\lim_{k \to \infty} a_k \neq 0$ then $\sum_{k=0}^{\infty} a_k$ diverges (Test for Divergence Theorem)

The Integral Test

- One of the tests for convergence
- Integral test: If f is continuous, decreasing, positive function on $[1, \infty)$ then $\sum_{k=1}^{\infty} f(k)$ converges iff $\sum_{1}^{\infty} f(x)\,\mathrm{d}x$ converges
  - The infinite sum is basically a Riemann sum with width 1
    - For a right hand sum $\sum_{k=2}^{n} f(k) \leq \int_{1}^{n} f(x)\,\mathrm{d}x$ since the function is decreasing
    - For a left hand sum $\int_{1}^{n} f(x)\,\mathrm{d}x \leq \sum_{k=1}^{n-1} f(k)$
  - If the integral converges then the right hand sum must also converge since it is less than the integral
  - If the integral diverges then the left hand sum must also diverge since it is greater than the integral
  - $\int_{1}^{\infty} f(x)\,\mathrm{d}x \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_{1}^{\infty} f(x)\,\mathrm{d}x$
  - The left boundary doesn't have to be 1
  - Since this goes both ways we can also use it to prove the convergence of an improper integral
- Example: Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$
  - Positive decreasing series so we can apply the integral test
  - $\int_{1}^{\infty} \frac{1}{x}\,\mathrm{d}x$ diverges, so the harmonic series diverges
- Example: p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$

- $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converges iff $p > 1$
- By the integral test this series converges iff $p > 1$
- Note the lower bound does not have to be 1; only the part that goes to infinity matters for convergence/divergence

Estimating Sums

- Define the *remainder* as $R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$
- By comparing the remainder to a Riemann sum we get $\int_{n+1}^\infty f(x)\,\mathrm{d}x \le R_n \le \int_n^\infty f(x)\,\mathrm{d}x$
  - $S_n + \int_{n+1}^\infty f(x)\,\mathrm{d}x \le S \le S_n + \int_n^\infty f(x)\,\mathrm{d}x$
- Example: $\sum_{n=1}^\infty \frac{1}{n^2}$
  - $S_5 = 1.463611$
  - $R_5 \le \int_5^\infty \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{5}$
  - $R_5 \ge \int_6^\infty \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{6}$
  - $S_5 + \frac{1}{6} \le S \le S_5 + \frac{1}{5}$
  - $1.63028 \le S \le 1.66361$

Basic Comparison Test

- Basic Comparison Test: Given $\sum a_k, \sum b_k; a_k > 0, b_k > 0$, then
  1. If $\sum b_k$ is convergent and $a_k \le b_k$ for all k sufficiently large then $\sum a_k$ converges
     - Proof: Define $S_n = \sum_{k=1}^n a_k, t_k = \sum_{k=1}^n b_k$
       * $t = \sum_{k=1}^\infty b_k$ exists
       * $\{ S_n \}$ is increasing since $a_k > 0$ as the sequences are both positive
       * $S_n \le t_n \le t$ so $\{ S_n \}$ is bounded above
       * By the monotonic sequence theorem $\sum a_k$ converges
  2. If $\sum b_k$ is divergent and $a_k \ge b_k$ for all k sufficiently large then $\sum a_k$ diverges
- Example: $\sum_{n=1}^\infty \frac{7}{17n^2 + 3n^{\frac{1}{2}} + 5}$
  - $17n^2 + 3n^{\frac{1}{2}} + 5 > 17n^2$ for $n \ge 1$ so $\frac{7}{17n^2 + 3n^{\frac{1}{2}} + 5} \le \frac{7}{17n^2}$
  - $\frac{7}{17}\sum \frac{1}{n^2}$ is convergent so this series converges
- Example: $\sum_{n=1}^\infty \frac{\ln\left(\frac{n}{1000}\right)}{n}$
  - Find k such that $\frac{\ln\left(\frac{k}{1000}\right)}{k} \ge \frac{1}{k}$
  - For k greater than this value we can do a comparison test to $\sum \frac{1}{n}$ to get that this series diverges