Lecture 13, Feb 8, 2022

Recursive Sequences

- How do we find the limit of a recursive sequence?
- First prove that it exists
- Example: $a_1 = 1, a_n = \sqrt{6 + a_{n-1}}$
  - We can show that it is increasing and has an upper bound of 3 using induction, so by the monotonic sequence theorem it converges
  - Knowing that the limit exists we can treat $\lim_{n\to\infty} a_n = L$ as a number, and $\lim_{n\to\infty} a_{n-1} = L$ also holds
  - Taking the limit of both sides $a_n = \sqrt{6 + a_{n-1}} \implies L = \sqrt{6 + L}$, solving for L yields 3 and -2 but the latter can't be a solution because all terms are positive

Series

- Infinite sums can lead to finite answers; e.g. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$
- Define a partial sum as: $S_n = \sum_{k=0}^{n} a_k$
- We can form a sequence of partial sums $\{ S_n \} = \{ a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots \}$
  - If this sequence does converge $\lim_{n\to\infty} S_n = L$ then we *define* $\sum_{k=0}^{\infty} a_k = L$
- Example: $\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} = \sum_{k=0}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3} \right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{1}{2} - \frac{1}{n+3} \implies \lim_{n\to\infty} S_n = \frac{1}{2}$
  - Telescoping series
- Geometric series: $\sum_{k=0}^{\infty} x^n$
  - Note x^0 is usually written as 1 even if $x = 0$
  - Sum is given by $\frac{1}{1-x}$ for $|x| < 1$ (diverges for $|x| \geq 1$)
  - Proof:
    * $S_n = 1 + x + \cdots + x^n$
    * $xS_n = x + x^2 + \cdots + x^{n+1}$
    * $S_n - xS_n = S_n(1-x) = 1 - x^{n+1}$
    * Assume $x \neq 1$, then $S_n = \frac{1 - x^{n+1}}{1 - x} \implies \lim_{n\to\infty} S_n = \frac{1}{1-x}$
      - Convergence requirement $|x| \geq 1$ comes from the limit of x^n
    * When $x = 1$, $S_n = 1 + 1 + \cdots = n + 1$ which diverges
    * When $x = -1$, $S_n = 1 - 1 + 1 - \cdots$ which is 1 for odd n and 0 for even n so it diverges
  - Example: $x = \frac{1}{2} \implies 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$
- Example: Repeating decimal $0.\overline{285714}$
  - $0.\overline{285714} = \frac{285714}{10^6} + \frac{285714}{10^{12}} + \cdots = \frac{285714}{10^6}\left(1 + \frac{1}{10^6} + \cdots\right) = \frac{285714}{10^6}\left(\frac{1}{1 - \frac{1}{10^6}}\right) = \frac{285714}{999999} = \frac{2}{7}$
- Example: $\frac{x}{4 - x^2}$ for $|x| < 2$, convert to geometric series
  - $\frac{x}{4 - x^2} = \frac{x}{4}\left(\frac{1}{1 - \frac{x^2}{4}}\right) = \frac{x}{4}\sum_{k=0}^{\infty}\left(\frac{x^2}{4}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+1} = \frac{1}{2}\left(\frac{x}{2} + \left(\frac{x}{2}\right)^3 + \cdots\right)$