Lecture 12, Feb 7, 2021

Limits of a Sequence

- It's often easier to work with the ordinary continuous function than the sequence itself; but note not all features of a function carries over to the sequence, e.g. discontinuities can make an unbounded function produce a bounded sequence
- We're really only interested in $\lim_{n\to\infty} a_n$, because it makes no sense to consider the limit at any finite value since we only get discrete points and no values around those points for a limit to make any sense
- We also can't differentiate or integrate a sequence so often taking the limit at infinity is more or less the only thing we can do
- Definition: $\lim_{n\to\infty} a_n = L \iff \forall\varepsilon > 0, \exists k > 0 \in \mathbb{Z} \ni n \geq k \implies |a_n - L| < \varepsilon$
- Example: Prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$
  - $\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| < \varepsilon \implies |n+1| > \frac{1}{\varepsilon}$
  - Choose $k = \frac{1}{\varepsilon}$ then $\left|\frac{n}{n+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{k} = \varepsilon$
- Theorem: Limits of sequences are unique: $\lim_{n\to\infty} a_n = L \land \lim_{n\to\infty} a_n = M \implies L = M$
- Definition: If a sequence has a limit, then it is *convergent*, otherwise it is *divergent*
  - Convergent sequences are always bounded (but bounded sequences aren't always convergent, e.g. $a_n = \cos(\pi n) = \{-1, 1, -1, 1, \cdots\}$ bounded above by 1 and below by -1 but still divergent)
  - Contrapositive: Unbounded sequences are always divergent
- Monotonic Sequence Theorem: A bounded non-decreasing sequence converges to its least upper bound; a bounded non-increasing sequence converges to its greatest lower bound
  - Include non-increasing and non-decreasing since a constant sequence is convergent
  - Monotonic only required for large n; the sequence can bounce around before that
  - Proven in Stewart (Chapter 11.1, Theorem 12, page 772)
- Sequence limit laws/theorems:
  - Limits are unique
  - Limit of sum/product/quotient to sum/product/quotient of limits
  - Limit of constant times thing is equal to constant times limit of thing
  - Limit of reciprocal is equal to the reciprocal of the limit assuming the limit does not equal zero
  - Pinching theorem/squeeze theorem: $a_n \leq b_n \leq c_n$ for large n and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$
    * Example: $\lim_{n\to\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{n}$ compare to $-\frac{1}{n}$ and $\frac{1}{n}$, since both go to 0 this sequence also goes to zero
- Theorem: Given $\lim_{n\to\infty} c_n = c$, if f is continuous at c then $\lim_{x\to\infty} f(c_n) = f(c)$
  - Example: $a_n = \sin\left(\frac{1}{n^2+1}\right)$: $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ and sin is continuous at 0 so $\lim_{n\to\infty} \sin\left(\frac{1}{n^2+1}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n^2+1}\right) = \sin 0 = 0$

Important Infinite Sequence Limits

1. For $x > 0$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$
   - $x^{\frac{1}{n}} = e^{\ln x^{\frac{1}{n}}} \to \ln x^{\frac{1}{n}} = \frac{1}{n}\ln x$ and $\lim_{n\to\infty} \frac{\ln x}{n} = 0$, so $\lim_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = e^0 = 1$
   - this relies on the exponential being continuous at 0
2. $|x| < 1 \implies \lim_{n\to\infty} x^n = 0$
   - $|x|^{n+1} < |x|^n$ since $|x| < 1$
   - Need to show $|x^n| < \varepsilon$ for all $n > k$

- $|x^n| = |x|^n < \varepsilon \implies |x| < \varepsilon^{\frac{1}{n}}$
- From 1, $\lim_{n\to\infty} \varepsilon^{\frac{1}{n}} = 1 > |x|$, therefore for sufficiently large k we will have $|x| < \varepsilon^{\frac{1}{k}}$
- Therefore $|x^n| < \varepsilon$ for all $n > k$

3. $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ for positive α
   - $0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha$
   - Take odd positive integer p such that $\frac{1}{p} < \alpha$ (e.g. $\alpha = 0.01$, p can be 101)
   - $0 < \left(\frac{1}{n}\right)^\alpha < \left(\frac{1}{n}\right)^{\frac{1}{p}}$
   - Since $x^{\frac{1}{p}} = \sqrt[p]{x}$ is continuous when p is an odd positive integer, $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{p}} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0$
   - By the squeeze theorem the limit is 0
4. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for $x \in \mathbb{R}$ (i.e. factorials grow faster than exponentials) and $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (i.e. factorials grow slower than n^n)
5. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
   - Use l'Hopital's rule to get $\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$
   - The denominator can be to any power
6. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$
   - $\ln n^{\frac{1}{n}} = \frac{1}{n} \ln n \implies \lim_{n\to\infty} n^{\frac{1}{n}} = 0$
   - $\lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = e^0 = 1$
7. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
   - For $x = 0$ this is satisfied trivially
   - $\ln\left(1 + \frac{x}{n}\right)^n = n \ln\left(1 + \frac{x}{n}\right) = \frac{x \ln\left(1 + \frac{x}{n}\right)}{\frac{x}{n}} = x\left(\frac{\ln\left(1 + \frac{x}{n}\right) - \ln 1}{\frac{x}{n}}\right)$
   - $\lim_{n\to\infty} \frac{\ln\left(1 + \frac{x}{n}\right) - \ln 1}{\frac{x}{n}} = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = 1$
   - Therefore $\lim_{x\to\infty} \ln\left(1 + \frac{x}{n}\right)^n = x \implies \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$