Lecture 11, Feb 4, 2022

Areas in Polar Coordinates

- Let $r = \rho(\theta)$ and $\alpha \leq \theta \leq \beta$; to find the area we can consider small slides that are sectors of a circle
  - Consider a small $\Delta\theta$; we can approximate the area by a circular sector: $\Delta A = \pi a^2 \cdot \frac{a\Delta\theta}{2\pi} = \frac{1}{2}a^2\Delta\theta$
  - Therefore area is given by $A = \frac{1}{2}\int_\alpha^\beta [\rho(\theta)]^2 \, \mathrm{d}\theta$
- Example: $r = 1 - \cos\theta$ (a cardioid oriented along the x axis, to the left of the y axis)
  - $$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, \mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, \mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} (\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)) \, \mathrm{d}\theta \\ &= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\ &= \frac{3}{2}\pi \end{aligned}$$
- For an area that lies between two polar curves we simply have $A = \frac{1}{2}\int_\alpha^\beta [\rho_1^2 - \rho_2^2] \, \mathrm{d}\theta$
- Example: $r^2 = 4\cos(2\theta)$ (Lemniscate oriented along the x axis) and $r = 1$
  - Lemniscate only exists for $-\frac{\pi}{4} \leq r \leq \frac{\pi}{4}$ due to the root
  - Find intersections: $4\cos(2\theta) = 1^2 \implies \theta = \pm 0.659\text{rad}$ or $\pi \pm 0.659\text{rad}$
  - Use symmetry
  - $A = \int_{-0.659}^{0.659} (4\cos(2\theta) - 1) \, \mathrm{d}\theta$

Arc Lengths in Polar Coordinates

- Parameterize the curve with θ and then use the parametric formula: $\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$
- $$\begin{aligned} s &= \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta \\ &= \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2} \, \mathrm{d}\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta \end{aligned}$$

Sequences

- A sequence is just a special function where the domain is limited to usually the positive integers (sometimes zero is included, and rarely negative numbers)
- Example: $f(x) = \frac{1}{x}$ is a function, $f(n) = a_n = \frac{1}{n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ is a sequence
  - Curly brackets usually denote a sequence
- A sequence can be *bounded* above, below, or not at all
  - e.g. $\left\{\frac{1}{n}\right\}$ is bounded above by 1, below by 0

- Sequences are collections of numbers; non-numbers such as infinity can't be part of a sequence
- Similar to function definitions, $\{ a_n \}$ is:
  - Increasing iff $a_n < a_{n+1}$
  - Non-decreasing iff $a_n \leq a_{n+1}$
  - Decreasing iff $a_n > a_{n+1}$
  - Non-increasing iff $a_n \geq a_{n+1}$
  - A sequence satisfying any of these is called a *monotonic sequence*
- Example: $a_n = 2^n \implies \frac{a_{n+1}}{a_n} = 2 > 1 \implies a_{n+1} > a_n$ so the sequence is monotonically increasing
  - To prove that this sequence is unbounded we follow a process similar to a limit at infinity
  - Find k such that $a_k > M$ for any M: $2^k > M \implies k > \frac{\ln M}{\ln 2}$
  - Since we've already shown that the sequence is increasing, $a_k > M \implies a_m > M$ if $m > k$