Lecture 10, Feb 1, 2022

Polar Coordinates

- $[r, \theta]$ instead of (x, y); if $r > 0$, point is $|r|$ from origin on ray of angle θ; if $r < 0$, point is $|r|$ from origin on ray of angle $\theta + \pi$
- Polar coordinates are not unique:
  1. The pole or origin $[0, \theta]$ for all θ
  2. $[r, \theta] = [r, \theta + 2\pi n]$ for $n \in \mathbb{Z}$
  3. $[r, \theta] = [-r, \theta + (2n+1)\pi]$ for $n \in \mathbb{Z}$
- Transformation: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
- In reverse $\begin{cases} r = \pm\sqrt{x^2+y^2} \\ \theta = \arctan\dfrac{y}{x} \end{cases}$ (be careful about the range of arctan)
  - Example: $(1, 2) = [\sqrt{5}, 1.107]$, $(1, -2) = [\sqrt{5}, -1.107]$, but since the range of arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, the point $(-1, 2) = [-\sqrt{5}, -1.107]$
- Useful when system has some kind of rotational symmetry
- Example:
  - Lines through the origin: $\theta = \alpha$
  - Vertical line $x = a \implies r\cos\theta = a \implies r = a\sec\theta$
  - Horizontal line $y = b \implies r\sin\theta = b \implies r = b\csc\theta$
  - General line $ax + by + c = 0 \implies r(a\cos\theta + b\sin\theta) + c = 0$
- Example: $r = 6\sin\theta$
  - $r^2 = 6r\sin\theta \implies x^2 + y^2 = 6y \implies x^2 + y^2 - 6y + 9 = 9 \implies x^2 + (y-3)^2 = 9$
- Symmetry about x axis is given by $[r, \theta] \to [r, -\theta]$; symmetry about y is given by $[r, \theta] \to [r, \pi - \theta]$; symmetry about origin is $[r, \pi + \theta]$

Graphing in Polar Coordinates

- Example: $r = \frac{1}{2} + \cos\theta$
  - $0 \le \theta \le 2\pi$ due to the periodic nature
  - Find values of θ that make $r = 0$: $\cos = -\frac{1}{2} \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$
  - Local max and min values of r: $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta = 0 \implies \theta = 0, \pi, 2\pi$; at $0, 2\pi \implies r = \frac{3}{2}$; at $\pi \implies r = -\frac{1}{2}$
  - Values on the x/y axis: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ is in the direction of the y axis; $r = \frac{1}{2}$
  - Symmetry: $\frac{1}{2} + \cos(-\theta) = \frac{1}{2}\cos\theta$, symmetry about x; $\frac{1}{2} + \cos(\pi - \theta) \neq \frac{1}{2} + \cos\theta$, not symmetric about y; $\frac{1}{2}\cos(\pi + \theta) \neq \frac{1}{2}\cos\theta$, not symmetric about the origin
  - Break the curve into sections based on where $r = 0$
  - Interval $\theta \in [0, \frac{2\pi}{3}]$
    * r positive, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta < 0$, so radius starts at $\frac{3}{2}$ and decreases to 0 at $\theta = \frac{2\pi}{3}$
  - This shape is called a Limaçon and this one has an inner loop
- Types of curves:
  1. Circles, e.g. $r = -2\cos\theta$ is a circle of radius 1 on the left of the y axis
     - cos – x axis crosses it, sin – y axis crosses it
       * Can also think of this as cos is symmetric about $\theta \to -\theta$ so the circle must lie on the x axis

  - Negative coefficient swaps from right of axis to left of axis or top of axis to bottom of axis
  - Coefficient out front is 2 times the radius
2. Cardioids: $r = a + a\cos\theta$
  - Same thing with the orientation for sine or cosine
  - Special case of a Limaçon, looks like a heart shape
3. Limaçons: $r = a + b\sin\theta$
  - Same thing with orientation
  - $a > b$ means we have a heart shape that never quite touches the origin
  - $a < b$ means we get an inner loop as the curve crosses the origin
4. Lemniscates: $r^2 = a\sin(2\theta)$
  - Only exists for particular values of θ due to the r^2
  - Has 2 petals, sine oriented on the $\theta = \frac{\pi}{4}$ line, cosine oriented along the x axis
5. Petal curves: $r = a\sin(n\theta)$ for n a positive integer
  - n petals for odd n, $2n$ petals for even n
  - Think of the $n = 1$ case – this is a circle, so only 1 petal, so for odd n we have n petals; in the other case we get $2n$ petals
  - This is because for an odd number of petals when we go from 0 to 2π we draw over the entire shape twice, but with even petals there is no overlap

Intersection of Polar Curves

- Mostly straightforward but we need to watch out for a few things
- Example: $\begin{cases} r = \sin\theta \\ r = -\cos\theta \end{cases}$
  - $\sin\theta = -\cos\theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$
  - $x = r\cos\theta = -\cos^2\frac{3\pi}{4} = -\frac{1}{2}$
  - $y = r\sin\theta = \sin^2\frac{3\pi}{4} = \frac{1}{2}$
  - Similarly at $\frac{7\pi}{4}$ we get $\left(-\frac{1}{2}, \frac{1}{2}\right)$
  - Notice these two points of intersection give us the same point
- Usually we should try to sketch the curve out
  - In the above example $r = \sin\theta$ is a circle tangent to the x axis and $r = -\cos\theta$ is a circle tangent to the y axis to the left
  - We do get two points of intersection but one is the origin – what did we do wrong?
  - Both curves actually go through the origin, but they do so at different values of θ!
  - We hit $\left(-\frac{1}{2}, \frac{1}{2}\right)$ because both of these circles are overlapping twice over $[0, 2\pi]$
- To check that two curves intersect at the origin, we need to check whether both curves go through the origin at some point for any θ

Tangents to Polar Curves

- First parameterize the curve: $r = r(\theta) \implies x = r(\theta)\cos\theta, y = r(\theta)\sin\theta$
- Recall for parametric curves the slope of the tangent is $\frac{y'}{x'}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$