Lecture 1, Prerecorded

Hyperbolic Trig Functions

- Hyperbolic sine: $\sinh x = \dfrac{e^x - e^{-x}}{2}$, hyperbolic cosine: $\cosh x = \dfrac{e^x + e^{-x}}{2}$
  - $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$ and $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$
  - Note there is no longer a negative sign!
- sinh is an odd function and has a point of inflection at the origin; for large positive it behaves like $\frac{1}{2}e^x$, large negative $-\frac{1}{2}e^{-x}$
  - Basically two exponentials stitched together
  - Note it does cross the origin ($\sinh 0 = 0$)
- cosh is an even function and always concave up; for large positive $\frac{1}{2}e^x$, large negative $\frac{1}{2}e^{-x}$
  - Does not cross the origin; $\cosh 0 = 1$
- Pythagorean identity analogue: $\cosh^2 x - \sinh^2 x = 1$
  - This means we can define the functions using a hyperbola
  - Notice a circle is $x^2 + y^2 = 1$ where points are $(\cos t, \sin t)$ and the area of the circular section is $\frac{1}{2}t$
  - Likewise a hyperbola is $x^2 - y^2 = 1$ where points are $(\cosh t, \sinh t)$ and the area of the hyperbolic sector is also $\frac{1}{2}t$
- Best known application is the shape of a catenary $y = a \cosh\left(\dfrac{x}{a}\right) + C$
- Define $\tanh = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
  - Other hyperbolic functions follow
  - Hyperbolic derivatives are extremely similar to regular trig derivatives; e.g. $\dfrac{\mathrm{d}}{\mathrm{d}x} \tanh x = \operatorname{sech}^2 x$
- Hyperbolic trig functions are *not* periodic
- We can find inverses:

$$\begin{aligned}
x &= \sinh y = \frac{e^y - e^{-y}}{2} \\
\implies 2x &= e^y - e^{-y} \\
\implies 0 &= e^y - e^{-y} - 2x \\
\implies 0 &= e^{2y} - 2xe^y - 1 \\
\implies e^y &= \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\
\implies e^y &= x + \sqrt{x^2 + 1} \\
\implies y &= \ln\left(x + \sqrt{x^2 + 1}\right) \\
\implies \sinh^{-1} x &= \ln\left(x + \sqrt{x^2 + 1}\right)
\end{aligned}$$

- $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \dfrac{1}{\sqrt{x^2 + 1}} \implies \displaystyle\int \frac{1}{\sqrt{1 + x^2}} \, \mathrm{d}x = \sinh^{-1} x + C = \ln\left(x + \sqrt{x^2 + 1}\right) + C$
- Important identities and differences:
  - sinh is odd, cosh is even
  - Pythagorean identity now uses minus; so does $1 - \tanh^2 x = \operatorname{sech}^2 x$
  - $\cosh(x + y)$ is two terms added instead of difference of two terms
  - $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$, without the minus sign
  - $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech} x = -\operatorname{sech} x \tanh x$ (as opposed to the normal $\dfrac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x$
  - Inverses: $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right)$

- Derivatives of inverses:
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \dfrac{1}{\sqrt{x^2+1}}$
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \tanh^{-1} x = \dfrac{1}{1-x^2}$
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{csch}^{-1} x = -\dfrac{1}{|x|\sqrt{x^2+1}}$
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech}^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}}$
    * $\dfrac{\mathrm{d}}{\mathrm{d}x} \coth^{-1} x = \dfrac{1}{1-x^2}$
    * Note the derivatives of $\tanh^{-1}$ and $\coth^{-1}$ have identical expressions, but the domain of each is different; $\tanh^{-1} x$ is defined for $|x| > 1$ and $\coth^{-1} x$ for $|x| > 1$