Lecture 1, Prerecorded

Hyperbolic Trig Functions

- Hyperbolic sine: $\sinh x = \frac{e^x - e^{-x}}{2}$, hyperbolic cosine: $\cosh x = \frac{e^x + e^{-x}}{2}$
  - $\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$ and $\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$
  - Note there is no longer a negative sign!
- sinh is an odd function and has a point of inflection at the origin; for large positive it behaves like $\frac{1}{2}e^x$, large negative $-\frac{1}{2}e^{-x}$
  - Basically two exponentials stitched together
  - Note it does cross the origin ($\sinh 0 = 0$)
- cosh is an even function and always concave up; for large positive $\frac{1}{2}e^x$, large negative $\frac{1}{2}e^{-x}$
  - Does not cross the origin; $\cosh 0 = 1$
- Pythagorean identity analogue: $\cosh^2 x - \sinh^2 x = 1$
  - This means we can define the functions using a hyperbola
  - Notice a circle is $x^2 + y^2 = 1$ where points are $(\cos t, \sin t)$ and the area of the circular section is $\frac{1}{2}t$
  - Likewise a hyperbola is $x^2 - y^2 = 1$ where points are $(\cosh t, \sinh t)$ and the area of the hyperbolic sector is also $\frac{1}{2}t$
- Best known application is the shape of a catenary $y = a \cosh\left(\frac{x}{a}\right) + C$
- Define $\tanh = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
  - Other hyperbolic functions follow
  - Hyperbolic derivatives are extremely similar to regular trig derivatives; e.g. $\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = \operatorname{sech}^2 x$
- Hyperbolic trig functions are *not* periodic
- We can find inverses:

$$\begin{aligned}
& x = \sinh y = \frac{e^y - e^{-y}}{2} \\
\implies & 2x = e^y - e^{-y} \\
\implies & 0 = e^y - e^{-y} - 2x \\
\implies & 0 = e^{2y} - 2xe^y - 1 \\
\implies & e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\
\implies & e^y = x + \sqrt{x^2 + 1} \\
\implies & y = \ln\left(x + \sqrt{x^2 + 1}\right) \\
\implies & \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)
\end{aligned}$$

- $\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}} \implies \int \frac{1}{\sqrt{1 + x^2}} \mathrm{d}x = \sinh^{-1} x + C = \ln\left(x + \sqrt{x^2 + 1}\right) + C$
- Important identities and differences:
  - sinh is odd, cosh is even
  - Pythagorean identity now uses minus; so does $1 - \tanh^2 x = \operatorname{sech}^2 x$
  - $\cosh(x + y)$ is two terms added instead of difference of two terms
  - $\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$, without the minus sign
  - $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech} x = -\operatorname{sech} x \tanh x$ (as opposed to the normal $\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x$
  - Inverses: $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$

- Derivatives of inverses:
  * $\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$
  * $\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$
  * $\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}x = \frac{1}{1-x^2}$
  * $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch}^{-1}x = -\frac{1}{|x|\sqrt{x^2+1}}$
  * $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}^{-1}x = -\frac{1}{x\sqrt{1-x^2}}$
  * $\frac{\mathrm{d}}{\mathrm{d}x}\coth^{-1}x = \frac{1}{1-x^2}$
  * Note the derivatives of $\tanh^{-1}$ and $\coth^{-1}$ have identical expressions, but the domain of each is different; $\tanh^{-1}x$ is defined for $|x| > 1$ and $\coth^{-1}x$ for $|x| > 1$

Lecture 2, Jan 11, 2022

Indeterminate Forms and L'Hopital's Rule

- Theorem: L'Hopital's Rule $(\frac{0}{0})$: if $f(x), g(x) \to 0$ as $x \to c$ (or a one sided or infinite limit) and $\frac{f'(x)}{g'(x)} \to L$, then $\frac{f(x)}{g(x)} = L$
  - We can apply it multiple times if the derivative limit ends up being 0/0 again, e.g. $\lim_{x\to 0}\frac{e^x - x - 1}{3x^2}$ requires using L'Hopital's Rule twice
- Only apply in the case of an indeterminate 0/0! If direct substitution gives us an answer, still using L'Hopital's Rule leads to an incorrect result; i.e. we can't use it to simplify limits

Proof of L'Hopital's Rule

- Cauchy Mean Value Theorem: Given f and g differentiable over (a, b) and continuous over $[a, b]$ and $g' \neq 0$ on (a, b), then $\exists r \in (a, b)$ s.t. $\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)}$
  - Form a special function to apply Rolle's Theorem on, similarly to how we previously proved the regular MVT
    * Recall Rolle's theorem: $g(a) = g(b) = 0 \implies \exists r \in (a, b)$ s.t. $g'(r) = 0$ (continuity and differentiability required)
  - Let $G(x) = [g(b) - g(a)][f(x) - f(a)] - [g(x) - g(a)][f(b) - f(a)] \implies G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)]$
  - Note $G(a) = 0 = G(b)$, therefore by Rolle's theorem $\exists r$ s.t. $G'(r) = [g(b) - g(a)]f'(r) - g'(r)[f(b) - f(a)] = 0 \implies \frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)}$
    * Note we know $g(b) - g(a) \neq 0$ because by MVT, $g(b) - g(a) = 0 \implies \exists c \in (a, b)$ s.t. $g'(c) = 0$ but we said $g' \neq 0$ on this interval
- Given as $x \to c^+ \implies f(x), g(x) \to 0$, $\frac{f'(x)}{g'(x)} \to L$, we need to prove $\frac{f(x)}{g(x)} \to L$
  - Consider the interval $[c, c + h]$, apply the Cauchy MVT (note if we wanted to prove $x \to c^-$, we can change the bounds here)
  - By Cauchy MVT, $\exists c_2$, $\frac{f'(c_2)}{g'(c_2)} = \frac{f(c+h) - f(c)}{g(c+h) - g(c)} = \frac{f(c+h)}{g(c+h)}$ (since $f(c) = g(c) = 0$)
  - Take the limit $h \to 0$, LHS: $\lim_{h\to 0}\frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} = L$, RHS: $\lim_{h\to 0}\frac{f(c+h)}{g(c+h)} = \lim_{x\to c^+}\frac{f(x)}{g(x)}$

- Therefore $\lim_{x \to c^+} \frac{f(x)}{g(x)} = L$
- We can make a similar argument for $x \to c^-$, which completes the two-sided limit
- Note this proof relies on $f(c) = g(c) = 0$, i.e. $0/0$ indeterminate form
- To extend this to $x \to \pm\infty$, let $x = \frac{1}{t}$ and take $t \to 0$

Other Indeterminate Forms

- Theorem: L'Hopital's Rule ($\frac{\infty}{\infty}$): if $f(x), g(x) \to \pm\infty$ as $x \to c$ (or a one sided or infinite limit) and $\frac{f'(x)}{g'(x)} \to L$, then $\frac{f(x)}{g(x)} = L$
  - Proof is a little more nuanced; not covered
  - Example: $\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \to \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \to \infty} \frac{\ln x}{x} = 0$
    - Any positive power of x grows faster than the logarithm
    - Likewise $\lim_{x \to \infty} \frac{x^m}{e^x} = 0$ for any positive integer m (keep applying L'Hopital's Rule)
  - Can also be applied multiple times
  - Notation: use $\stackrel{*}{=}$ to denote equality for the limits due to L'Hopital's Rule
- For $\infty \cdot 0$, we can rearrange this
  - Example: $\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}}$ now substitution gets us the $\frac{\infty}{\infty}$ form which we can use L'Hopital's Rule on
- For forms such as $0^0, \infty^0, 1^\infty$, we can take the exponential of the log of both sides
  - This relies on the exponential and logarithm being continuous functions; this means we can apply the function and bring the limit on the argument to the entire function
  - Example: 0^0 indeterminate form: $\lim_{x \to 0} x^x$
    - $\lim_{x \to 0} x^x = \lim_{x \to 0} e^{\ln x^x} = \lim_{x \to 0} e^{x \ln x}$
    - Since e^x is continuous, we bring the limit to $\lim_{x \to 0} x \ln x$
      - Direct substitution results in $0 \cdot \infty$, another indeterminate form, but $\frac{\ln x}{\frac{1}{x}}$ is an $\frac{\infty}{\infty}$ indeterminate type, so L'Hopital's Rule can be used
    - $\lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \stackrel{*}{=} \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$
    - Substituting back in, we get that $\lim_{x \to 0} x^x = e^0 = 1$
  - Example: ∞^0 form: $\lim_{x \to \infty} (x+2)^{\frac{2}{\ln x}}$
    - $\lim_{x \to \infty} (x+2)^{\frac{2}{\ln x}} = \lim_{x \to \infty} e^{\ln(x+2)^{\frac{2}{\ln x}}} = \lim_{x \to \infty} \frac{2\ln(x+2)}{\ln x}$
    - Apply L'Hopital's Rule twice to get 2

Lecture 3, Jan 14, 2022

Integration By Parts

- The integral counterpart to the product rule

- $$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)\\ \implies &\int \left(f(x)g'(x) + g(x)f'(x)\right)\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x)\,\mathrm{d}x\\ \implies &\int f(x)g'(x)\,\mathrm{d}x + \int g(x)f'(x)\,\mathrm{d}x = f(x)g(x)\\ \implies &\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x\end{aligned}$$
- Let $u = f(x), v = g(x) \implies \mathrm{d}u = f'(x)\,\mathrm{d}x, \mathrm{d}v = g'(x)\,\mathrm{d}x \implies \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$
- Example: $$\begin{aligned}\int x\sin x\,\mathrm{d}x &= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x\\ &= x(-\cos x) - \int 1\cdot(-\cos x)\,\mathrm{d}x\\ &= -x\cos x + \sin x + C\end{aligned}$$
- Integration by parts should simplify the integral; choose $f(x)$ so it becomes simpler when differentiated, and $g(x)$ so that it is easy to integrate
- Example: $$\begin{aligned}&\int \ln x\,\mathrm{d}x\\ \implies &x\ln x - \int x\cdot\frac{1}{x}\,\mathrm{d}x\\ \implies &x\ln x - x + C\end{aligned}$$
- Integration by parts can be applied multiple times if the resulting integral is still not quite simple
- Sometimes we get back the original integral: $$\begin{aligned}\int e^x\sin x\,\mathrm{d}x &= -e^x\cos x + \int e^x\cos x\,\mathrm{d}x\\ &= -e^x\cos x + e^x\sin x - \int e^x\sin x\,\mathrm{d}x\\ \implies 2\int e^x\sin x\,\mathrm{d}x &= e^x(\sin x - \cos x) + C'\\ \implies \int e^x\sin x\,\mathrm{d}x &= \frac{1}{2}e^x(\sin x - \cos x) + C\end{aligned}$$
  - Remember when subtracting $\int f(x)\,\mathrm{d}x - \int f(x)\,\mathrm{d}x \neq 0$, because there is still an integration constant!
- For definite integrals, $\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,\mathrm{d}x$
  - The first term there is a boundary term
- Example: $$\begin{aligned}\int_0^1 \tan^{-1}x\,\mathrm{d}x &= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x\\ &= \frac{\pi}{4} - \frac{1}{2}\int_{t=0}^{t=1}\frac{1}{u}\,\mathrm{d}u\\ &= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1\\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2\end{aligned}$$
- Using integration by parts, we can prove reduction formulas:
  - $\int \sin^n x\,\mathrm{d}x = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,\mathrm{d}x,\ n \geq 2$

- $$\begin{aligned}\int \sin^n x \,\mathrm{d}x &= \sin^{n-1} x \sin x \,\mathrm{d}x \\ &= -\cos x \sin^{n-1} x - \int (n-1)\sin^{n-2} x \cos x \cdot -\cos x \,\mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \,\mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,\mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \,\mathrm{d}x - (n-1)\int \sin^n x \,\mathrm{d}x \end{aligned}$$
$$\implies n\int \sin^n x \,\mathrm{d}x = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \,\mathrm{d}x$$
$$\implies \int \sin^n x \,\mathrm{d}x = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x \,\mathrm{d}x$$

Lecture 4, Jan 17, 2022

Trigonometric Integrals

Powers of Sines and Cosines

- When we have powers of sines and cosines, try to convert it to only one sin and the rest cos or one cos and the rest sin to u-sub
- Even powers can be converted using $\sin^2 x + \cos^2 x = 1$
  - Example: $$\begin{aligned}\int \cos^3 x \,\mathrm{d}x &= \int \cos^2 x \cos x \,\mathrm{d}x \\ &= \int (1-\sin^2 x)\cos x \,\mathrm{d}x \\ &= \int (1-u^2)\,\mathrm{d}u \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C\end{aligned}$$
- For odd powers, separate a single factor and convert the remaining even power
  - Example: $$\begin{aligned}\int \sin^5 x \cos^2 x \,\mathrm{d}x &= \int (1-\cos^2 x)^2 \sin x \cos^2 x \,\mathrm{d}x \\ &= -\int (1-u^2)^2 u^2 \,\mathrm{d}u \\ &= -\frac{u^3}{3} + 2\frac{2u^5}{5} - \frac{u^7}{7} + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C\end{aligned}$$
- Half-angle identities: $$\begin{aligned} &\cos(2x) = \cos^2 x - \sin^2 x \text{ and} && \cos(2x) = \cos^2 x - \sin^2 x \\ \implies &\cos(2x) + 1 = 2\cos^2 x && \implies 1 - \cos(2x) = 2\sin^2 x \\ \implies &\cos^2 x = \frac{1}{2}(1+\cos(2x)) && \implies \sin^2 x = \frac{1}{2}(1-\cos(2x))\end{aligned}$$
  - Example: $$\begin{aligned}\int_0^\pi \sin^2 x \,\mathrm{d}x &= \frac{1}{2}\int_0^\pi 1 - \cos(2x)\,\mathrm{d}x \\ &= \left[\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)\right]_0^\pi \\ &= \frac{1}{2}\pi\end{aligned}$$

- Example: $\sin^4 x \,\mathrm{d}x = \int (\sin^2 x)^2 \,\mathrm{d}x$

$$
\begin{aligned}
&= \int \left(\frac{1}{2}(1-\cos(2x))\right)^2 \,\mathrm{d}x \\
&= \frac{1}{4}\int \big(1+\cos^2(2x)-2\cos(2x)\big) \,\mathrm{d}x \\
&= \frac{1}{4}\int \left(1+\frac{1}{2}(1+\cos(4x))-2\cos(2x)\right) \,\mathrm{d}x \\
&= \frac{1}{4}\int \left(\frac{3}{2}+\frac{1}{2}\cos(4x)-2\cos(2x)\right) \,\mathrm{d}x \\
&= \frac{1}{4}\left(\frac{3}{2}x-\sin(2x)+\frac{1}{8}\sin(4x)\right)+C \\
&= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C
\end{aligned}
$$

- Remember $\sin x \cos x = \frac{1}{2}\sin(2x)$
- Summary of strategy for $\int \sin^m x \cos^n x \,\mathrm{d}x$:
  - If m is odd, save one sine factor and convert the rest to cosine; likewise for n
  - If both powers are even, use the half angle identities

Powers of Secant and Tangent

- Similar to sine and cosine since $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$; use identity $1+\tan^2 x = \sec^2 x$
- Remember $\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \tan x \sec x$
- Example: $\int \tan^6 x \sec^4 x \,\mathrm{d}x = \int \tan^6 x \sec^2 x(1+\tan^2 x)\,\mathrm{d}x$

$$
\begin{aligned}
&= \int u^6(1+u^2)\,\mathrm{d}u \\
&= \frac{u^7}{7}+\frac{u^9}{9}+C \\
&= \frac{1}{7}\tan^7 x+\frac{1}{9}\tan^9 x+C
\end{aligned}
$$

- Example: $\int \tan^5 x \sec^7 x \,\mathrm{d}x = \int \tan^4 x \sec^6 x \sec x \tan x \,\mathrm{d}x = \int (\sec^2 x-1)^2 \sec^6 x \sec x \tan x \,\mathrm{d}x$

$$
\begin{aligned}
&= \int (u^2-1)^2 u^6 \,\mathrm{d}u \\
&= \frac{u^11}{11}-2\frac{u^9}{9}+\frac{u^7}{7}+C \\
&= \frac{1}{11}\sec^1 1x-\frac{2}{9}\sec^9 x+\frac{1}{7}\sec^7 x+C
\end{aligned}
$$

- Remember $\int \tan x \,\mathrm{d}x = \ln|\sec x|+C$ and $\int \sec x \,\mathrm{d}x = \ln|\sec x+\tan x|+C$
- For $\int \tan^m x \sec^n x \,\mathrm{d}x$:
  - For even power of sec, save one $\sec^2 x$, convert the rest to tangent, and substitute $u=\tan x$
  - For odd power of tan, save one tan for a $\tan x \sec x$, cover the rest of $\tan^2$ to secant, then substitute $u=\sec x \implies \mathrm{d}u = \tan x \sec x \,\mathrm{d}x$

- We may have to integrate by parts; example: $\int \sec^3 x \, \mathrm{d}x = \sec x \tan x - \int (\sec x \tan x) \tan x \, \mathrm{d}x$
$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, \mathrm{d}x$$
$$= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \ln|\sec x + \tan x| + C$$
  - Now we can solve for $\int \sec^3 x \, \mathrm{d}x$ since it appears on both sides

Product-To-Sum

- Integrals of the form $\int \sin(mx)\cos(nx) \, \mathrm{d}x$, $\int \sin(mx)\sin(nx) \, \mathrm{d}x$ or $\int \cos(mx)\cos(nx) \, \mathrm{d}x$ can be solved using the product-to-sum identities
- $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$
- $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$
- $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

Trigonometric Substitution

- Inverse substituting x for a trig function could simplify radicals of some forms
- $\sqrt{a^2 - x^2}$: Substitute $x = a\sin\theta \implies \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$
$$= a\sqrt{1 - \sin^2\theta}$$
$$= a|\cos\theta|$$
- $\sqrt{a^2 + x^2}$: Substitute $x = a\tan\theta \implies \sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta}$
$$= a\sqrt{1 + \tan^2\theta}$$
$$= a|\sec\theta|$$
- $\sqrt{x^2 - a^2}$: Substitute $x = a\sec\theta \implies \sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta + a^2}$
$$= a\sqrt{\sec^2\theta - 1}$$
$$= a|\tan\theta|$$
  - Alternatively, use the hyperbolic substitution $x = \cosh^2 t \implies \sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)}$
$$= a\sinh t$$
- Example: $\int \frac{\sqrt{9 - x^2}}{x^2} \, \mathrm{d}x = \int \frac{3\cos\theta}{9\sin^2\theta} 2\cos\theta \, \mathrm{d}\theta$
$$= \int \cot^2\theta \, \mathrm{d}\theta$$
$$= \int (\csc^2\theta - 1) \, \mathrm{d}\theta$$
$$= -\cot\theta - \theta + C$$
  - To return this to x, $x = 3\sin\theta \implies \sin\theta = \frac{x}{3} \implies \theta = \sin^{-1}\left(\frac{x}{3}\right)$; draw a triangle with the right sides and we get $\cot\theta = \frac{\sqrt{9 - x^2}}{x}$
  - For definite integrals we can just change the limits of integration to avoid having to convert back
- Example: Find the area of the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$
  - By symmetry we only need to worry about one quadrant where $x, y \geq 0$, so solving for y we have $y = \frac{b}{a}\sqrt{a^2 - x^2}$

- Substitute $x = a\sin\theta \implies \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$; change the bounds of integration: $x = a\sin\theta = 0 \implies \theta = 0, x = a\sin\theta = a \implies \sin\theta = 1 \implies \theta = \frac{\pi}{2}$
- $$\begin{aligned} A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\mathrm{d}x &= 4\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\,\mathrm{d}\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \cos^2\ \mathrm{d}\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos(2\theta))\,\mathrm{d}\theta \\ &= 2ab\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} \\ &= \pi ab \end{aligned}$$

Lecture 5, Jan 18, 2022

Partial Fraction Decomposition

- For $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of P is less than that of Q, we can write this as a sum of simpler polynomials
  - If the degree of P is equal to or greater than Q we must first perform long division and then decompose the remainder
- Factor Q into factors of $ax + b$ and $ax^2 + bx + c$ (linear and irreducible quadratic factors) and then express it as a sum of partial fractions: $\dfrac{A}{(ax+b)^i}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^i}$
- Cases below:
  1. Q is a product of unique linear factors: $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{a_1x + b_1} + \dfrac{A_2}{a_2x + b_2} + \cdots$
     - Example: $\dfrac{x^2 + 2x + 1}{2x^3 + 3x^2 - 2x} \to \dfrac{x^2 + 2x + 1}{x(2x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2}$
       * $x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$
       * We can now expand the right side and group terms to obtain a system of equations for A, B, C
       * Alternatively we can set x to convenient values
         - $x = 0 \implies -1 = -2A \implies A = \frac{1}{2}$
         - $x = \frac{1}{2} \implies \frac{1}{4} = \frac{5}{4}B \implies B = \frac{1}{5}$
         - $x = -2 \implies -1 = 10C \implies C = -\frac{1}{10}$
  2. Q is a product of non-unique linear factors: If $(ax+b)^r$ occurs, then instead of just $\dfrac{A}{ax+b}$, we have $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_r}{(ax+b)^r}$
     - Example: $\dfrac{4x}{x^3 - x^2 - x + 1} \to \dfrac{4x}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$
       * $4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$
       * We can once again expand and get the equations for the coefficients
       * We can still set convenient values of x:
         - $x = 1 \implies 4 = 2B \implies B = 2$
         - $x = -1 \implies -4 = 4C \implies C = -1$
         - Can't find an x that would cancel the other terms and leave A, but we can still obtain a relation for A
         - $x = 0 \implies 0 = -A + B + C \implies A = B + C = -1$

3. Q contains unique irreducible quadratic factors: Each quadratic factor corresponds to a term of $\dfrac{Ax+B}{ax^2+bx+c}$ in the expansion
   - Example: $\dfrac{2x^2-x+4}{x^3+4x} \to \dfrac{2x^2-x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$
     * $2x^2-x+4 = A(x^2+4)+(Bx+C)x$
     * Expand: $2x^2-x+4 = (A+B)x^2+Cx+4A \implies \begin{cases} A+B=2 \\ C=-1 \\ 4A=4 \end{cases} \implies \begin{cases} A=1 \\ B=1 \\ C=-1 \end{cases}$
     * Therefore $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{1}{x} + \dfrac{x-1}{x^2+4}$
   - To integrate the quadratic terms: $\displaystyle\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
   - In general to integrate $\dfrac{Ax+B}{ax^2+bx+c}$, complete the square $\dfrac{Ax+B}{(ex+f)^2+g}$ then make the substitution $u=ex+f$ so the fraction is now $\dfrac{Cu+D}{u^2+g}$, which can be split into $\dfrac{Cu}{u^2+g}$ (can use substitution) and $\dfrac{D}{u^2+g}$ (can use the expression above)
   - $$\begin{aligned} \int \frac{2x}{x^2+x+1}\,\mathrm{d}x &= \int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \frac{1}{x^2+x+1}\,\mathrm{d}x \\ &= \ln|x^2+x+1| + \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x \\ &= \ln|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+C \end{aligned}$$
4. Q contains repeated irreducible quadratic factors: Similar to 2, each factor corresponds to terms of $\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$
   - $\dfrac{Ax+B}{(x^2+\beta x+\gamma)^2} = \dfrac{A}{2}\left(\dfrac{2x+B}{(x^2+\beta x+\gamma)^2} + \dfrac{2\frac{B}{A}-B}{(x^2+\beta x+\gamma)^2}\right)$
     * The first can be directly solved using substitution, the second needs completing the square and then using trig substitution

- We can convert the irreducible quadratic terms into complex linear terms:
  - Example: $\displaystyle\int \frac{2x}{x^2+1}\,\mathrm{d}x$
    * Try $\dfrac{2x}{x^2+1} = \dfrac{A}{x+i} + \dfrac{B}{x-i}$
    * $2x = A(x-i)+B(x+i)$
    * $x=i \implies 2i = 2iB \implies B=1$ similarly $A=1$
    * Integrating we get $\ln|x+i|+\ln|x-i|+C$ and the arguments of ln multiply to get us x^2+1 back
  - $\ln(a+ib) = \ln\sqrt{a^2+b^2} + i\tan^{-1}\left(\dfrac{b}{a}\right)$
    * Example: $\displaystyle\int \frac{1}{x^2+1}\,\mathrm{d}x$
- Nonrational functions can be made rational by an appropriate substitution
  - Example: $\displaystyle\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$
    * Let $u=\sqrt{x+4}$ then $u^2=x+4 \implies x=u^2-4$ and $\mathrm{d}x = 2u\,\mathrm{d}u$ so $\displaystyle\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x = \int \frac{2u^2}{u^2-4}\,\mathrm{d}u = 2\int \left(1+\frac{4}{u^2-4}\right)\mathrm{d}u$

- Weierstrass substitution: $t = \tan\dfrac{x}{2}$
  - This allows us to convert ratios of sines and cosines to rational functions
  - $\sin x = \dfrac{2t}{1+t^2}, \cos x = \dfrac{1-t^2}{1+t^2}$
  - $\mathrm{d}x = \dfrac{2}{1+t^2}\,\mathrm{d}t$
  - Example: $$\begin{aligned}\int \frac{1}{1+\cos x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int \frac{2}{(1+t^2)+(1-t^2)}\,\mathrm{d}t \\ &= \int \mathrm{d}t \\ &= t + C \\ &= \tan\left(\frac{x}{2}\right) + C\end{aligned}$$

Lecture 6, Jan 24, 2022

Improper Integrals

- Remember: Infinity is NaN, so we must define any expression that contains it
- Definition: If $\lim\limits_{b\to\infty} = \displaystyle\int_a^b f(x)\,\mathrm{d}x = L$, define $\displaystyle\int_0^\infty f(x)\,\mathrm{d}x = L$; these are called *improper integrals*
  - We can also have the lower limit go to infinity in the same way, or both bounds be infinite
  - Also define $[f(x)]_a^\infty$ as $\lim\limits_{b\to\infty}[f(x)]_a^b$ if the limit exists/converges
  - If the limit doesn't exist then we say that the integral *diverges*
- Example: $$\begin{aligned}\int_0^\infty e^{-x}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b \\ &= \lim_{b\to\infty}(1-e^{-b}) \\ &= 1\end{aligned}$$
- Not all improper integrals converge! Example: $$\begin{aligned}\int_3^\infty \frac{1}{x}\,\mathrm{d}x &= \lim_{b\to\infty}[\ln x]_3^b \\ &= \lim_{b\to\infty}(\ln b - \ln 3) \\ &= \infty\end{aligned}$$
- Integrals can diverge for other reasons: $$\begin{aligned}\int_{-\infty}^{2\pi}\sin x\,\mathrm{d}x &= \lim_{a\to\infty}\int_a^{2\pi}\sin x\,\mathrm{d}x \\ &= \lim_{a\to\infty}[-\cos x]_a^{2\pi} \\ &= \lim_{a\to\infty}(-1+\cos a)\end{aligned}$$
  - Since $\cos a$ does not approach any value for $a\to\infty$ this integral is undefined
- General example: $\displaystyle\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ for $p > 0, p \neq 1$ and $a > 0$

- $$\begin{aligned}\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x &= \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}x^{-p+1}\right]_a^b \\ &= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{1-p}-\frac{a^{-p+1}}{1-p}\right) \\ &= \frac{a^{1-p}}{p-1} && \text{for } p>1\end{aligned}$$
  - If $p<1$ then this will diverge
- Technique to demonstrate convergence: given f, g continuous and $0 \le f(x) \le g(x)$ for $x \in [a, \infty]$, then if $\int_a^\infty g\,\mathrm{d}x$ converges so does $\int_a^\infty f\,\mathrm{d}x$; similarly if $\int_a^\infty f\,\mathrm{d}x$ diverges then so does $\int_0^\infty g\,\mathrm{d}x$
  - Example: $\int_2^\infty \frac{1}{\sqrt{1+x^{\frac{44}{17}}}}\,\mathrm{d}x$
    * We note $\frac{1}{\sqrt{1+x^{\frac{44}{17}}}} < \frac{1}{\sqrt{x^{\frac{44}{17}}}} = \frac{1}{x^{\frac{22}{17}}}$
    * Since $\frac{22}{17} > 1$, $\int_2^\infty \frac{1}{x^{\frac{22}{17}}}\,\mathrm{d}x$ converges
    * Since this is larger than our integrand, our integral will also converge
  - Example: $\int_3^\infty \frac{1}{(7+x^2)^{\frac{1}{2}}}\,\mathrm{d}x$
    * Note $(7+x^2)^{\frac{1}{2}} < \sqrt{7}+x$ for $x \ge 3$
    * Therefore $\frac{1}{(7+x^2)^{\frac{1}{2}}} > \frac{1}{\sqrt{7}+x}$
    * Since $\int_3^\infty \frac{1}{\sqrt{7}+x}\,\mathrm{d}x$ diverges and our integrand is always greater than this integrand, our integral also diverges
- When we have both bounds infinite we can break it up: $\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$
  - a can be anything here but we can usually choose it to be something convenient
  - Note $\int_{-\infty}^\infty \neq \lim_{b\to\infty}\int_{-b}^b f(x)\,\mathrm{d}x$
    * This works if the integral converges because in that case it doesn't matter how fast we approach infinity; however if the integral diverges this will give us the wrong answer
    * Example: $\int_{-\infty}^\infty x\,\mathrm{d}x$
      - If we break this up we can see this integral obviously doesn't converge
      - But if $$\begin{aligned}\int_{-\infty}^\infty x\,\mathrm{d}x &= \lim_{b\to\infty}\int_{-b}^b x\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left(\frac{b^2}{2}-\frac{b^2}{2}\right) \\ &= 0\end{aligned}$$
      - We get zero because we happen to approach the two limits at the same rate
      - If instead $$\begin{aligned}\lim_{b\to\infty}\int_{-b}^{2b} x\,\mathrm{d}x &= \lim_{b\to\infty}\left(\frac{4b^2}{2}-\frac{b^2}{2}\right) \\ &= \infty\end{aligned}$$
        - If this integral was one-sided it wouldn't matter at what rate we approach infinity
- We can also have improper integrals where the interval contains a discontinuity
  - Suppose $\lim_{x\to b^-} f(x) = \infty$ then we can still define $\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x$
- When the discontinuity is in the middle, break up the integral at the discontinuity; both pieces need to

converge for the improper integral to converge

  - If we have a discontinuity at z, then $\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to z^-}\int_a^c f(x)\,\mathrm{d}x + \lim_{c\to z^+}\int_c^b f(x)\,\mathrm{d}x$
  - We need to be careful because if we just plugged in the numbers as if there was no discontinuity we would get the wrong answer
  - Example: $\int_{-1}^{3}\frac{1}{x^2}\,\mathrm{d}x$
    * If we evaluate it as $\left[-\frac{1}{x}\right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$, which makes no sense as this integral should diverge and the function is always positive so we should never get a negative area
  - When we have an integral we need to make sure the integrand has no discontinuity over the region; if it does then we need to treat it as an improper integral
- For the interval 0 to 1, the $\frac{1}{x^p}$ rule is the reverse; if $p < 1$ then the integral converges, otherwise it diverges

Lecture 7, Jan 25, 2022

Arc Length

- Consider a curve $y = f(x)$ for $x \in [a, b]$, where $y'(x)$ exists; how do we define the length of the curve?
  - Approximate the curve by more and more finer straight line segments
  - With Δx between segments the length of each segment is $s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$
  - We may simplify this using the MVT; recall that $\frac{\Delta y_i}{\Delta x_i} = y'(x_i^*)$ for some $x_i^* \in [x_i, x_i + \Delta x_i]$ so $\Delta y_i = y'(x_i^*)\Delta x_i$
  - $s_i = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - Convert to integral for total length: $s = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$
- Example: $f(x) = x^{\frac{3}{2}}$ for $x \in [0, 44]$
  - $f'(x) = \frac{3}{2}x^{\frac{1}{2}} \implies s = \int_0^{44}\sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x$

$$= \left[\frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{\frac{1}{2}}\right]_0^{44}$$

$$= 296$$

- Arc length integrals can be messy due to the square root; analytic solutions can only be found for a small set of functions

Surfaces of Revolution

- Side surface area of a cut off cone: $A = \pi(r + R)l$ where r is the smaller radius, R is the bigger radius and l is the slant height
- Rotate the curve $f(x)$ around the x axis to generate a surface; how do we find this area?
  - Cut into pieces, each piece is a cut off cone so the area is $A_i = \pi(f(x_{i-1}) + f(x_i))s_i = \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - Use IVT to simplify $f(x_{i-1}) + f(x_i)$: there exists some x_i^{**} such that $f(x_{i-1}) + f(x_i) = 2f(x_i^{**})$
  - $A_i = 2\pi f(x_i^{**})\sqrt{1 + (f'(x_i^*))^2}\Delta x_i$
  - $A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x$
- Example: $f(x) = \sqrt{x}$ over $x \in [0, 1]$

- $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \implies A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,\mathrm{d}x$

$$= \pi\int_0^1 \sqrt{4x+1}\,\mathrm{d}x$$

$$= \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$

- Example: $f(x) = \frac{1}{x}$ over $x \in [1, \infty)$
  - Finite volume, infinite surface area (Gabriel's horn)
  - We can use a comparison test to demonstrate that this integral diverges

Applications to Physics and Engineering

Hydrostatic Pressure

- When an object is submerged in liquid it experiences a pressure force always perpendicular to the surface; the magnitude of force is $\rho g d$ where ρ is the density, g is the gravitational constant and d is the depth
- Consider a plate submerged vertically (x axis going down with 0 at the surface of the water)
  - $F_i = w(x_i^*)\Delta x_i \cdot \rho g x_i^*$ (area term times force term)
  - Taking the limit we get the integral $F = \int_a^b \rho g x w(x)\,\mathrm{d}x$
- Example: Circular water main with 1 meter radius; if we cap the pipe when its half filled with water, how much force will be pushing on the cap?
  - Since the pipe is half filled take the middle of the pipe as $x = 0$
  - Width of the pipe is $2\sqrt{1-x^2}$ by Pythagorean theorem
  - $F = 2\int_0^1 \rho g x\sqrt{1-x^2}\,\mathrm{d}x$

$$= 2\rho g\left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\rho g$$

$$= \frac{2}{3}\mathrm{m}^3 \cdot 1000\mathrm{kg/m}^3 \cdot 9.8\mathrm{m/s}^2$$

$$= 6533\mathrm{N}$$

Lecture 8, Jan 28, 2022

Moments and Centres of Mass

- To find the centroid, make use of two principles:
  1. Symmetry: $(\bar{x}, \bar{y})$ must be on any axis of symmetry
     - In simple cases where we have multiple axes of symmetry we can just use the intersection of the axes to find the centroid
  2. Additivity: The centroid of a bigger piece is a weighted average of centroids of smaller pieces (where the weights are the areas of the smaller pieces)
     - $\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \cdots}{A_1 + A_2 + \cdots}$ similarly for $\bar{y}$
- To extend this to a more complicated region we break any region into fine rectangles and take the limit to get an integral
  - Each piece has area $A_i = f(x_i^*)\Delta x_i$
  - $\bar{x}_i = x_i^*$
  - $\bar{y}_i = \frac{1}{2}f(x_i^*)$

- Take the limit and $\bar{x} = \dfrac{\int_a^b xf(x)\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x}$, $\bar{y} = \dfrac{\frac{1}{2}\int_a^b f(x)^2\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x}$
- If we have a region bounded by two curves, use the additivity rule and subtract the smaller function
  - $\bar{x}A + \bar{x}_g A_g = \bar{x}_f A_f \implies \bar{x}A = \bar{x}_f A_f - \bar{x}_g A_g$
- Example: Region between $f(x) = 6, g(x) = 3$ between $x \in [2, 5]$
  - $\bar{x}A = \displaystyle\int_2^5 x(6-3)\,\mathrm{d}x = 3\left[\frac{1}{2}x^2\right]_2^5 = \frac{63}{2} \implies \bar{x} = \frac{7}{2}$
  - $\bar{y}A = \displaystyle\int_2^5 \frac{1}{2}(36-9)\,\mathrm{d}x = \frac{81}{2} \implies \bar{y} = \frac{9}{2}$
- Pappus' Theorem on Volumes: For a solid of revolution $V = 2\pi\bar{R}A$ where A is the area of the region being rotated and $\bar{R}$ is distance from the axis of rotation to the centroid of the region
  - Example: Elliptical torus with cross section as an ellipse with area $A = \pi ab$, radius is the major radius R so $V = 2\pi^2 Rab$
  - This theorem is equivalent to doing the integrals
  - Consider a washer method about x: $V_x = \displaystyle\int_a^b \pi\left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x = 2\pi\int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x = 2\pi\bar{y}A$
  - Consider the shell method about y: $V_y = \displaystyle\int_a^b 2\pi x(f(x) - g(x))\,\mathrm{d}x = 2\pi\int_a^b x(f(x) - g(x))\,\mathrm{d}x = 2\pi\bar{x}A$

Parametric Curves

- We can describe curves by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ where we introduced the parameter t and made it the new independent variable
- Example: Projectile motion in physics
- Example: Line between two points: $\begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \end{cases}$
- Parametric representations inherently contain more information; e.g. parameterizing projectile motion introduces information about time/velocity while $y(x)$ by itself only contains information about trajectory shape
- Intersection of two curves happens when $y_1(x) = y_2(x)$; collision happens when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$
  - To solve for collision, it can be helpful to solve for intersections first
- Example: $\begin{cases} x_1(t) = 2t + 6 \\ y_1(t) = 5 - 4t \end{cases}, \begin{cases} x_2(t) = 3 - 5\cos(\pi t) \\ y_2(t) = 1 + 5\sin(\pi t) \end{cases}, t \geq 0$
  - Intersection:
    * Curve 1: $t = \dfrac{x-6}{2} \implies y_1(x) = 17 - 2x, x \geq 6$
    * Curve 2: $(x-3)^2 + (y-1)^2 = 25$
    * Solving for intersections yields $(6, 5)$ and $(8, 1)$
  - Collision:
    * $(6, 5)$ happens on curve 1 at $t = 0$; curve 2 is at $(-2, 1)$ at this point, so there is no collision
    * $(8, 1)$ happens on curve 1 at $t = 1$; curve 2 is at $(8, 1)$ at this point, so there is a collision

Lecture 9, Jan 31, 2022

Calculus with Parametric Curves

Tangents

- Since these curves can be more complex than functions there are more cases for tangents:

  1. Ordinary tangent
  2. No tangent (sharp point)
  3. Multiple tangents (due to the curve intersecting itself)
- To find the tangent we can again take the limit of a secant line
  - $m = \dfrac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} = \dfrac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}}$
  - Take the limit $h \to 0$ this just becomes $m = \dfrac{y'(t_0)}{x'(t_0)}$
    * Note we're differentiating with respect to t
  - The tangent line equation is $y'(t_0)(x - x_0) - x'(t_0)(y - y_0) = 0$
    * $\dfrac{y - y_0}{x - x_0} = \dfrac{y'(t_0)}{x'(t_0)} \implies y'(t_0)(x - x_0) = x'(t_0)(y - y_0) \implies y'(t_0)(x - x_0) - x'(t_0)(y - y_0) = 0$
- $x'(t_0) = 0$ gives a vertical tangent, $y'(t_0) = 0$ gives a horizontal tangent
  - If both are zero then we get no information
- Tangents can be used for curve sketching
  - Find derivatives of $x(t)$ and $y(t)$ and find locations of vertical and horizontal tangents
  - Also calculate slope at locations such as the origin and other points of interest

Areas

- Formula for area under parametric curve between $x(t_1)$ and $x(t_2)$ is just $A = \displaystyle\int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t$ (essentially a substitution)
- To calculate the area inside a closed curve, direction starts mattering; define the positive traversal direction such that the enclosed area is always on the left (i.e. counterclockwise)
- Going from t_1 to t_5 in the positive direction, the enclosed area is $A = -\displaystyle\int_{t_1}^{t_5} y(t)x'(t)\,\mathrm{d}t$, or $A = \displaystyle\int t_1{}^{t_5} x(t)y'(t)\,\mathrm{d}t$
  - Note the need for the negative sign when integrating with x but not y
  - Note the values of x and y at t_1 and t_5 are equal but $t_1 \neq t_5$
- Example: Ellipse $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$
  - Curve repeats $\theta \in [0, 2\pi]$
  - $x' = -a\sin\theta \implies A = -\displaystyle\int_0^{2\pi} -ab\sin^2\theta\,\mathrm{d}\theta = ab\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \pi ab$
  - We could also have done it with y' and get the same result

Arc Length

- Arc length is now given by $\sum \sqrt{\Delta x^2 + \Delta y^2}$; in the limit we get $s = \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$
  - Note if we let $x = t$, $y = f(t) = f(x)$ we get back the arc length formula for functions
- Example: $\begin{cases} x = \theta\cos\theta \\ y = \theta\sin\theta \end{cases}, \theta \in [0, 2\pi]$

- $s = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\,\mathrm{d}\theta$
  $= \int_0^{2\pi} \sqrt{1+\theta^2}\,\mathrm{d}\theta$
  $= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\theta + \sqrt{1+\theta^2}\right|\right]_0^{2\pi}$
  $= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right)$

Surface Area

- Formula remains the same: $A = \int_a^b 2\pi y\,\mathrm{d}s$
  - Recall in parametric form $\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$

Lecture 10, Feb 1, 2022

Polar Coordinates

- $[r, \theta]$ instead of (x, y); if $r > 0$, point is $|r|$ from origin on ray of angle θ; if $r < 0$, point is $|r|$ from origin on ray of angle $\theta + \pi$
- Polar coordinates are not unique:
  1. The pole or origin $[0, \theta]$ for all θ
  2. $[r, \theta] = [r, \theta + 2\pi n]$ for $n \in \mathbb{Z}$
  3. $[r, \theta] = [-r, \theta + (2n+1)\pi]$ for $n \in \mathbb{Z}$
- Transformation: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
- In reverse $\begin{cases} r = \pm\sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$ (be careful about the range of arctan)
  - Example: $(1, 2) = [\sqrt{5}, 1.107]$, $(1, -2) = [\sqrt{5}, -1.107]$, but since the range of arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, the point $(-1, 2) = [-\sqrt{5}, -1.107]$
- Useful when system has some kind of rotational symmetry
- Example:
  - Lines through the origin: $\theta = \alpha$
  - Vertical line $x = a \implies r\cos\theta = a \implies r = a\sec\theta$
  - Horizontal line $y = b \implies r\sin\theta = b \implies r = b\csc\theta$
  - General line $ax + by + c = 0 \implies r(a\cos\theta + b\sin\theta) + c = 0$
- Example: $r = 6\sin\theta$
  - $r^2 = 6r\sin\theta \implies x^2 + y^2 = 6y \implies x^2 + y^2 - 6y + 9 = 9 \implies x^2 + (y-3)^2 = 9$
- Symmetry about x axis is given by $[r, \theta] \to [r, -\theta]$; symmetry about y is given by $[r, \theta] \to [r, \pi - \theta]$; symmetry about origin is $[r, \pi + \theta]$

Graphing in Polar Coordinates

- Example: $r = \frac{1}{2} + \cos\theta$
  - $0 \le \theta \le 2\pi$ due to the periodic nature
  - Find values of θ that make $r = 0$: $\cos = -\frac{1}{2} \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$
  - Local max and min values of r: $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta = 0 \implies \theta = 0, \pi, 2\pi$; at $0, 2\pi \implies r = \frac{3}{2}$; at $\pi \implies r = -\frac{1}{2}$

- Values on the x/y axis: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ is in the direction of the y axis; $r = \frac{1}{2}$
  - Symmetry: $\frac{1}{2} + \cos(-\theta) = \frac{1}{2}\cos\theta$, symmetry about x; $\frac{1}{2} + \cos(\pi - \theta) \neq \frac{1}{2} + \cos\theta$, not symmetric about y; $\frac{1}{2}\cos(\pi + \theta) \neq \frac{1}{2}\cos\theta$, not symmetric about the origin
  - Break the curve into sections based on where $r = 0$
  - Interval $\theta \in [0, \frac{2\pi}{3}]$
    * r positive, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta < 0$, so radius starts at $\frac{3}{2}$ and decreases to 0 at $\theta = \frac{2\pi}{3}$
  - This shape is called a Limaçon and this one has an inner loop
- Types of curves:
  1. Circles, e.g. $r = -2\cos\theta$ is a circle of radius 1 on the left of the y axis
     - cos – x axis crosses it, sin – y axis crosses it
       * Can also think of this as cos is symmetric about $\theta \to -\theta$ so the circle must lie on the x axis
     - Negative coefficient swaps from right of axis to left of axis or top of axis to bottom of axis
     - Coefficient out front is 2 times the radius
  2. Cardioids: $r = a + a\cos\theta$
     - Same thing with the orientation for sine or cosine
     - Special case of a Limaçon, looks like a heart shape
  3. Limaçons: $r = a + b\sin\theta$
     - Same thing with orientation
     - $a > b$ means we have a heart shape that never quite touches the origin
     - $a < b$ means we get an inner loop as the curve crosses the origin
  4. Lemniscates: $r^2 = a\sin(2\theta)$
     - Only exists for particular values of θ due to the r^2
     - Has 2 petals, sine oriented on the $\theta = \frac{\pi}{4}$ line, cosine oriented along the x axis
  5. Petal curves: $r = a\sin(n\theta)$ for n a positive integer
     - n petals for odd n, $2n$ petals for even n
     - Think of the $n = 1$ case – this is a circle, so only 1 petal, so for odd n we have n petals; in the other case we get $2n$ petals
     - This is because for an odd number of petals when we go from 0 to 2π we draw over the entire shape twice, but with even petals there is no overlap

Intersection of Polar Curves

- Mostly straightforward but we need to watch out for a few things
- Example: $\begin{cases} r = \sin\theta \\ r = -\cos\theta \end{cases}$
  - $\sin\theta = -\cos\theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$
  - $x = r\cos\theta = -\cos^2\frac{3\pi}{4} = -\frac{1}{2}$
  - $y = r\sin\theta = \sin^2\frac{3\pi}{4} = \frac{1}{2}$
  - Similarly at $\frac{7\pi}{4}$ we get $\left(-\frac{1}{2}, \frac{1}{2}\right)$
  - Notice these two points of intersection give us the same point
- Usually we should try to sketch the curve out
  - In the above example $r = \sin\theta$ is a circle tangent to the x axis and $r = -\cos\theta$ is a circle tangent to the y axis to the left
  - We do get two points of intersection but one is the origin – what did we do wrong?
  - Both curves actually go through the origin, but they do so at different values of θ!

- We hit $\left(-\frac{1}{2}, \frac{1}{2}\right)$ because both of these circles are overlapping twice over $[0, 2\pi]$
- To check that two curves intersect at the origin, we need to check whether both curves go through the origin at some point for any θ

Tangents to Polar Curves

- First parameterize the curve: $r = r(\theta) \implies x = r(\theta)\cos\theta, y = r(\theta)\sin\theta$
- Recall for parametric curves the slope of the tangent is $\frac{y'}{x'}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$

Lecture 11, Feb 4, 2022

Areas in Polar Coordinates

- Let $r = \rho(\theta)$ and $\alpha \leq \theta \leq \beta$; to find the area we can consider small slides that are sectors of a circle
  - Consider a small $\Delta\theta$; we can approximate the area by a circular sector: $\Delta A = \pi a^2 \cdot \frac{a\Delta\theta}{2\pi} = \frac{1}{2}a^2\Delta\theta$
  - Therefore area is given by $A = \frac{1}{2}\int_\alpha^\beta [\rho(\theta)]^2 \,\mathrm{d}\theta$
- Example: $r = 1 - \cos\theta$ (a cardioid oriented along the x axis, to the left of the y axis)
  - $$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi} (\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)) \,\mathrm{d}\theta \\ &= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\ &= \frac{3}{2}\pi \end{aligned}$$
- For an area that lies between two polar curves we simply have $A = \frac{1}{2}\int_\alpha^\beta [\rho_1^2 - \rho_2^2] \,\mathrm{d}\theta$
- Example: $r^2 = 4\cos(2\theta)$ (Lemniscate oriented along the x axis) and $r = 1$
  - Lemniscate only exists for $-\frac{\pi}{4} \leq r \leq \frac{\pi}{4}$ due to the root
  - Find intersections: $4\cos(2\theta) = 1^2 \implies \theta = \pm 0.659\text{rad}$ or $\pi \pm 0.659\text{rad}$
  - Use symmetry
  - $A = \int_{-0.659}^{0.659} (4\cos(2\theta) - 1) \,\mathrm{d}\theta$

Arc Lengths in Polar Coordinates

- Parameterize the curve with θ and then use the parametric formula: $\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$

- $s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$
  $= \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta\right)^2}\,\mathrm{d}\theta$
  $= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

Sequences

- A sequence is just a special function where the domain is limited to usually the positive integers (sometimes zero is included, and rarely negative numbers)
- Example: $f(x) = \frac{1}{x}$ is a function, $f(n) = a_n = \frac{1}{n} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \cdots \right\}$ is a sequence
  - Curly brackets usually denote a sequence
- A sequence can be *bounded* above, below, or not at all
  - e.g. $\left\{ \frac{1}{n} \right\}$ is bounded above by 1, below by 0
- Sequences are collections of numbers; non-numbers such as infinity can't be part of a sequence
- Similar to function definitions, $\{ a_n \}$ is:
  - Increasing iff $a_n < a_{n+1}$
  - Non-decreasing iff $a_n \leq a_{n+1}$
  - Decreasing iff $a_n > a_{n+1}$
  - Non-increasing iff $a_n \geq a_{n+1}$
  - A sequence satisfying any of these is called a *monotonic sequence*
- Example: $a_n = 2^n \implies \frac{a_{n+1}}{a_n} = 2 > 1 \implies a_{n+1} > a_n$ so the sequence is monotonically increasing
  - To prove that this sequence is unbounded we follow a process similar to a limit at infinity
  - Find k such that $a_k > M$ for any M: $2^k > M \implies k > \frac{\ln M}{\ln 2}$
  - Since we've already shown that the sequence is increasing, $a_k > M \implies a_m > M$ if $m > k$

Lecture 12, Feb 7, 2021

Limits of a Sequence

- It's often easier to work with the ordinary continuous function than the sequence itself; but note not all features of a function carries over to the sequence, e.g. discontinuities can make an unbounded function produce a bounded sequence
- We're really only interested in $\lim_{n\to\infty} a_n$, because it makes no sense to consider the limit at any finite value since we only get discrete points and no values around those points for a limit to make any sense
- We also can't differentiate or integrate a sequence so often taking the limit at infinity is more or less the only thing we can do
- Definition: $\lim_{n\to\infty} a_n = L \iff \forall\varepsilon > 0, \exists k > 0 \in \mathbb{Z} \ni n \geq k \implies |a_n - L| < \varepsilon$
- Example: Prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$
  - $\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| < \varepsilon \implies |n+1| > \frac{1}{\varepsilon}$
  - Choose $k = \frac{1}{\varepsilon}$ then $\left|\frac{n}{n+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{k} = \varepsilon$
- Theorem: Limits of sequences are unique: $\lim_{n\to\infty} a_n = L \wedge \lim_{n\to\infty} a_n = M \implies L = M$
- Definition: If a sequence has a limit, then it is *convergent*, otherwise it is *divergent*
  - Convergent sequences are always bounded (but bounded sequences aren't always convergent,

e.g. $a_n = \cos(\pi n) = \{ -1, 1, -1, 1, \cdots \}$ bounded above by 1 and below by -1 but still divergent)
  - Contrapositive: Unbounded sequences are always divergent
- Monotonic Sequence Theorem: A bounded non-decreasing sequence converges to its least upper bound; a bounded non-increasing sequence converges to its greatest lower bound
  - Include non-increasing and non-decreasing since a constant sequence is convergent
  - Monotonic only required for large n; the sequence can bounce around before that
  - Proven in Stewart (Chapter 11.1, Theorem 12, page 772)
- Sequence limit laws/theorems:
  - Limits are unique
  - Limit of sum/product/quotient to sum/product/quotient of limits
  - Limit of constant times thing is equal to constant times limit of thing
  - Limit of reciprocal is equal to the reciprocal of the limit assuming the limit does not equal zero
  - Pinching theorem/squeeze theorem: $a_n \leq b_n \leq c_n$ for large n and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$
    - Example: $\lim_{n\to\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{n}$ compare to $-\frac{1}{n}$ and $\frac{1}{n}$, since both go to 0 this sequence also goes to zero
- Theorem: Given $\lim_{n\to\infty} c_n = c$, if f is continuous at c then $\lim_{x\to\infty} f(c_n) = f(c)$
  - Example: $a_n = \sin\left(\frac{1}{n^2+1}\right)$: $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ and sin is continuous at 0 so $\lim_{n\to\infty} \sin\left(\frac{1}{n^2+1}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n^2+1}\right) = \sin 0 = 0$

Important Infinite Sequence Limits

1. For $x > 0$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$
   - $x^{\frac{1}{n}} = e^{\ln x^{\frac{1}{n}}} \to \ln x^{\frac{1}{n}} = \frac{1}{n}\ln x$ and $\lim_{n\to\infty} \frac{\ln x}{n} = 0$, so $\lim_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = e^0 = 1$
   - this relies on the exponential being continuous at 0
2. $|x| < 1 \implies \lim_{n\to\infty} x^n = 0$
   - $|x|^{n+1} < |x|^n$ since $|x| < 1$
   - Need to show $|x^n| < \varepsilon$ for all $n > k$
     - $|x^n| = |x|^n < \varepsilon \implies |x| < \varepsilon^{\frac{1}{n}}$
     - From 1, $\lim_{n\to\infty} \varepsilon^{\frac{1}{n}} = 1 > |x|$, therefore for sufficiently large k we will have $|x| < \varepsilon^{\frac{1}{k}}$
     - Therefore $|x^n| < \varepsilon$ for all $n > k$
3. $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ for positive α
   - $0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha$
   - Take odd positive integer p such that $\frac{1}{p} < \alpha$ (e.g. $\alpha = 0.01$, p can be 101)
   - $0 < \left(\frac{1}{n}\right)^\alpha < \left(\frac{1}{n}\right)^{\frac{1}{p}}$
   - Since $x^{\frac{1}{p}} = \sqrt[p]{x}$ is continuous when p is an odd positive integer, $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{p}} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0$
   - By the squeeze theorem the limit is 0
4. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for $x \in \mathbb{R}$ (i.e. factorials grow faster than exponentials) and $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (i.e. factorials grow slower than n^n)
5. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

- Use l'Hopital's rule to get $\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$
- The denominator can be to any power

6. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$
   - $\ln n^{\frac{1}{n}} = \frac{1}{n}\ln n \implies \lim_{n\to\infty} n^{\frac{1}{n}} = 0$
   - $\lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = e^0 = 1$
7. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
   - For $x = 0$ this is satisfied trivially
   - $\ln\left(1 + \frac{x}{n}\right)^n = n\ln\left(1 + \frac{x}{n}\right) = \frac{x\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}} = x\left(\frac{\ln\left(1+\frac{x}{n}\right) - \ln 1}{\frac{x}{n}}\right)$
   - $\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right) - \ln 1}{\frac{x}{n}} = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = 1$
   - Therefore $\lim_{x\to\infty} \ln\left(1 + \frac{x}{n}\right)^n = x \implies \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$

Lecture 13, Feb 8, 2022

Recursive Sequences

- How do we find the limit of a recursive sequence?
- First prove that it exists
- Example: $a_1 = 1, a_n = \sqrt{6 + a_{n-1}}$
  - We can show that it is increasing and has an upper bound of 3 using induction, so by the monotonic sequence theorem it converges
  - Knowing that the limit exists we can treat $\lim_{n\to\infty} a_n = L$ as a number, and $\lim_{n\to\infty} a_{n-1} = L$ also holds
  - Taking the limit of both sides $a_n = \sqrt{6 + a_{n-1}} \implies L = \sqrt{6 + L}$, solving for L yields 3 and -2 but the latter can't be a solution because all terms are positive

Series

- Infinite sums can lead to finite answers; e.g. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$
- Define a partial sum as: $S_n = \sum_{k=0}^{n} a_k$
- We can form a sequence of partial sums $\{\, S_n \,\} = \{\, a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots \}$
  - If this sequence does converge $\lim_{n\to\infty} S_n = L$ then we *define* $\sum_{k=0}^{\infty} a_k = L$
- Example: $\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} = \sum_{k=0}^{\infty}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{1}{2} - \frac{1}{n+3} \implies \lim_{n\to\infty} S_n = \frac{1}{2}$
  - Telescoping series
- Geometric series: $\sum_{k=0}^{\infty} x^n$
  - Note x^0 is usually written as 1 even if $x = 0$
  - Sum is given by $\frac{1}{1-x}$ for $|x| < 1$ (diverges for $|x| \geq 1$)
  - Proof:
    * $S_n = 1 + x + \cdots + x^n$
    * $xS_n = x + x^2 + \cdots + x^{n+1}$

        * $S_n - xS_n = S_n(1-x) = 1 - x^{n+1}$
        * Assume $x \neq 1$, then $S_n = \dfrac{1-x^{n+1}}{1-x} \implies \lim_{n\to\infty} S_n = \dfrac{1}{1-x}$
            - Convergence requirement $|x| \geq 1$ comes from the limit of x^n
        * When $x = 1$, $S_n = 1 + 1 + \cdots = n + 1$ which diverges
        * When $x = -1$, $S_n = 1 - 1 + 1 - \cdots$ which is 1 for odd n and 0 for even n so it diverges
    - Example: $x = \dfrac{1}{2} \implies 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots = \dfrac{1}{1-\frac{1}{2}} = 2$
- Example: Repeating decimal $0.\overline{285714}$
    - $0.\overline{285714} = \dfrac{285714}{10^6} + \dfrac{285714}{10^{12}} + \cdots = \dfrac{285714}{10^6}\left(1 + \dfrac{1}{10^6} + \cdots\right) = \dfrac{285714}{10^6}\left(\dfrac{1}{1-\frac{1}{10^6}}\right) = \dfrac{285714}{999999} = \dfrac{2}{7}$
- Example: $\dfrac{x}{4-x^2}$ for $|x| < 2$, convert to geometric series
    - $\dfrac{x}{4-x^2} = \dfrac{x}{4}\left(\dfrac{1}{1-\frac{x^2}{4}}\right) = \dfrac{x}{4}\sum_{k=0}^{\infty}\left(\dfrac{x^2}{4}\right)^k = \dfrac{1}{2}\sum_{k=0}^{\infty}\left(\dfrac{x}{2}\right)^{2k+1} = \dfrac{1}{2}\left(\dfrac{x}{2} + \left(\dfrac{x}{2}\right)^3 + \cdots\right)$

Lecture 14, Feb 11, 2022

Infinite Series Theorems

- Limit laws apply for infinite series
    - Sum rule
    - Scalar multiplication
- Theorem: $\sum_{k=0}^{\infty} a_k$ converges iff $\sum_{k=j}^{\infty} a_k$, i.e. convergence only depends on the behaviour of the part extending to infinity
    - Note by definition every term in a_k has to be finite since all terms in a sequence have to be numbers, so we can't have anything infinite in the first j terms
    - $\sum_{k=j}^{\infty} a_k = L - (a_0 + a_1 + \cdots + a_{j-1})$
- Example: Given $\sum_{k=4}^{\infty} \dfrac{3^{k-1}}{4^{3k+1}}$ converges, then we know $\sum_{k=0}^{\infty} \dfrac{3^{k-1}}{4^{3k+1}} = \dfrac{1}{12} + \dfrac{1}{64} + \dfrac{3}{47} + \dfrac{9}{40} + \sum_{k=4}^{\infty} \dfrac{3^{k-1}}{4^{3k+1}}$ also converges
- Theorem: If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$
    - Contrapositive: If $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=0}^{\infty} a_k$ diverges (Test for Divergence Theorem)

The Integral Test

- One of the tests for convergence
- Integral test: If f is continuous, decreasing, positive function on $[1, \infty)$ then $\sum_{k=1}^{\infty} f(k)$ converges iff $\sum_{1}^{\infty} f(x)\,\mathrm{d}x$ converges
    - The infinite sum is basically a Riemann sum with width 1
        * For a right hand sum $\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\,\mathrm{d}x$ since the function is decreasing

    * For a left hand sum $\int_1^n f(x)\,\mathrm{d}x \leq \sum_{k=1}^{n-1} f(k)$
  - If the integral converges then the right hand sum must also converge since it is less than the integral
  - If the integral diverges then the left hand sum must also diverge since it is greater than the integral
  - $\int_1^\infty f(x)\,\mathrm{d}x \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_1^\infty f(x)\,\mathrm{d}x$
  - The left boundary doesn't have to be 1
  - Since this goes both ways we can also use it to prove the convergence of an improper integral
- Example: Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$
  - Positive decreasing series so we can apply the integral test
  - $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges, so the harmonic series diverges
- Example: p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$
  - $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converges iff $p > 1$
  - By the integral test this series converges iff $p > 1$
  - Note the lower bound does not have to be 1; only the part that goes to infinity matters for convergence/divergence

Estimating Sums

- Define the *remainder* as $R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$
- By comparing the remainder to a Riemann sum we get $\int_{n+1}^\infty f(x)\,\mathrm{d}x \leq R_n \leq \int_n^\infty f(x)\,\mathrm{d}x$
  - $S_n + \int_{n+1}^\infty f(x)\,\mathrm{d}x \leq S \leq S_n + \int_n^\infty f(x)\,\mathrm{d}x$
- Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$
  - $S_5 = 1.463611$
  - $R_5 \leq \int_5^\infty \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{5}$
  - $R_5 \geq \int_6^\infty \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{6}$
  - $S_5 + \frac{1}{6} \leq S \leq S_5 + \frac{1}{5}$
  - $1.63028 \leq S \leq 1.66361$

Basic Comparison Test

- Basic Comparison Test: Given $\sum a_k, \sum b_k; a_k > 0, b_k > 0$, then
  1. If $\sum b_k$ is convergent and $a_k \leq b_k$ for all k sufficiently large then $\sum a_k$ converges
     - Proof: Define $S_n = \sum_{k=1}^{n} a_k, t_k = \sum_{k=1}^{n} b_k$
       * $t = \sum_{k=1}^{\infty} b_k$ exists
       * $\{ S_n \}$ is increasing since $a_k > 0$ as the sequences are both positive
       * $S_n \leq t_n \leq t$ so $\{ S_n \}$ is bounded above
       * By the monotonic sequence theorem $\sum a_k$ converges

2. If $\sum b_k$ is divergent and $a_k \geq b_k$ for all k sufficiently large then $\sum a_k$ diverges

- Example: $\sum_{n=1}^{\infty} \frac{7}{17n^2 + 3n^{\frac{1}{2}} + 5}$
  - $17n^2 + 3n^{\frac{1}{2}} + 5 > 17n^2$ for $n \geq 1$ so $\frac{7}{17n^2 + 3n^{\frac{1}{2}} + 5} \leq \frac{7}{17n^2}$
  - $\frac{7}{17} \sum \frac{1}{n^2}$ is convergent so this series converges
- Example: $\sum_{n=1}^{\infty} \frac{\ln\left(\frac{n}{1000}\right)}{n}$
  - Find k such that $\frac{\ln\left(\frac{k}{1000}\right)}{k} \geq \frac{1}{k}$
  - For k greater than this value we can do a comparison test to $\sum \frac{1}{n}$ to get that this series diverges

Lecture 15, Feb 14, 2022

Limit Comparison Test

- Limit Comparison Test: Given $\sum a_k, \sum b_k; a_k > 0, b_k > 0$, then
  1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ then both series converge or diverge
  2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ then convergence of $\sum b_n$ implies convergence of $\sum a_n$
  3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ then divergence of $\sum b_n$ implies divergence of $\sum a_n$
- Case 1 proof:
  - $\lim_{n \to \infty} \frac{a_n}{b_n} = c \implies \left|\frac{a_n}{b_n} - c\right| < \varepsilon$ for $n > N$
  - Choose $\varepsilon = \frac{c}{2} \implies \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \implies \frac{c}{2} b_n \leq a_n \leq \frac{3c}{2} b_n$ for $n \geq N$
  - If $\sum b_n$ converges then so does $\frac{3c}{2} \sum b_n$ so $\sum a_n$ converges by the comparison test
  - If $\sum b_n$ diverges then so does $\frac{c}{2}$ so $\sum a_n$ diverges by the comparison
- Example: $\frac{1}{n^3 - n}$
  - Limit comparison test to $\frac{1}{n^3}$
  - $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^3}{n^3 - n} = 1$ so by LCT, the series converges since $\sum \frac{1}{n^3}$ is convergent

Alternating Series

- Sometimes series contain both positive and negative terms
- An *alternating series* alternates between positive and negative terms
  - Not all series with both positive and negative terms are alternating, e.g. $\frac{\cos n}{n^2}$
- Alternating series usually have a $(-1)^n$ term to make the alternating signs
- Alternating Series Test: Let $\{ a_k \}$ be a sequence of positive numbers; if $a_{k+1} < a_k$ and $\lim_{k \to \infty} a_k = 0$, then $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges
  - Since the terms alternate between positive and negative and are decreasing we're always bouncing around in a range that's getting smaller
  - Any partial sum must lie between the two previous sums
  - Proof:

    * First look at the even terms: $S_2 = a_1 - a_2 > 0, S_4 = S_2 + (a_3 - a_4) > 0, \cdots, S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2}$
      - By induction, $\{ S_{2n} \}$ is a monotonically increasing sequence
      - Also, $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$; since all the terms after a_1 are positive, $S_{2n} < a_1$ for all n
      - Since $\{ S_{2n} \}$ is monotonic and bounded by the monotonic sequence theorem it converges
      - Let $\lim_{n\to\infty} S_{2n} = L$
    * Now look at the odd terms: $S_{2n+1} = S_{2n} + a_{2n+1}$
      - $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$
      - First limit is L as above, second limit is 0 since we require that the sequence goes to 0, therefore $\lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L$
    * Since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1} = L$, $\lim_{n\to\infty} S_n = L$ so the series converges
  - If $a_n \to 0$ is *not* true, then the series *always* diverges, but the series being monotonically decreasing is *not* a strict requirement for convergence
- Example: Alternating harmonic series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges because absolute value of terms decreases and magnitude goes to 0

Alternating Series Error Bounds

- The properties of an alternating series give us that L will always be between S_n and S_{n+1}: $|L - S_n| \leq a_{n+1}$
- The error in a partial sum is less than the next term in the series

Absolute and Conditional Convergence

- Definition: If $\sum |a_k|$ converges, then $\sum a_k$ is *absolutely convergent*; if $\sum a_k$ converges but not $\sum |a_k|$, then $\sum a_k$ is *conditionally convergent*
- Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ also converges
  - Proof: $-|a_n| \leq a_n \leq |a_n| \implies 0 \leq a_n + |a_n| \leq 2|a_n|$
    * Let $a_n + |a_n| = b_n \implies 0 \leq b_n \leq 2|a_n|$
    * We know $2\sum |a_n|$ converges, therefore $\sum b_n$ converges by the comparison test since $a_n + |a_n| \leq 2|a_n|$
    * Rearranging, $\sum a_n = \sum b_n - \sum |a_n|$
    * Because both $\sum b_n$ and $\sum |a_n|$ is convergent, $\sum a_n$ is convergent
- Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent
- $\sum \frac{(-1)^{k+1}}{k} = \sum \frac{1}{2k-1} - \sum \frac{1}{2k}$, but for conditionally convergent series we have an $\infty - \infty$ situation
  - This means we must be careful when moving the terms around; depending on the rate that both sums approach infinity we can get a different value out of it

Lecture 16, Feb 15, 2022

The Root and Ratio Tests

- The Root Test: Given $\sum a_k, a_k \geq 0$, if $\lim_{k\to\infty} a_k^{\frac{1}{k}} = \lim_{k\to\infty} \sqrt[k]{a_k} = p$, then
  1. If $p < 1$ then $\sum a_k$ converges
  2. If $p > 1$ then $\sum a_k$ diverges
  3. If $p = 1$ then the test is inconclusive

- Proof of part 1:
  - Given $p < 1$, choose μ such that $p < \mu < 1$
  - $a_k^{\frac{1}{k}} < \mu$ for sufficiently large k since it approaches p
  - $a_k < \mu^k$ for sufficiently large k
  - $\sum \mu^k$ converges because it is a geometric series with $x < 1$, therefore by the comparison test $\sum a_k$ converges
- In effect the root test is a limit comparison test with a geometric series
- Example: $\sum \left(\frac{n^2+1}{2n^2+1}\right)^n$
  - $a_n^{\frac{1}{n}} = \frac{n^2+1}{2n^2+1}$
  - $\lim_{n\to\infty} a_n = \frac{1}{2} < 1$ so the series converges
- The Ratio Test: Given $\sum a_k, a_k > 0$, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$, then
  1. If $\lambda < 1$ then $\sum a_k$ converges
  2. If $\lambda > 1$ then $\sum a_k$ diverges
  3. If $\lambda = 1$ then the test is inconclusive
- Proof of part 1:
  - Given $\lambda < 1$ choose μ such that $\lambda < \mu < 1$
  - Then $\frac{a_{k+1}}{a_k} < \mu$ for some sufficiently large $k > K$
  - $\frac{a_{K+1}}{a_K} < \mu \implies a_{K+1} < \mu a_K \implies a_{K+2} < \mu a_{K+1} < \mu^2 a_K \implies \cdots \implies a_{K+j} < \mu^j a_K$
  - Let $n = K + j \implies a_n < \mu^{n-K} a_K = \frac{a_K}{\mu^K}\mu^n$
  - $\sum a_n < \frac{a_K}{\mu^K}\sum \mu^n$, which is a convergent geometric series since $\mu < 1$, therefore $\sum a_n$ converges as well
- Example: $\sum \frac{k^2}{e^k}$
  - $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \frac{(k+1)^2}{k^2} \cdot \frac{1}{e}$
  - The ratio goes to $\frac{1}{e} < 1$ in the limit so the series is convergent
- Both tests give you absolute convergence

Power Series

- Definition: A *power series* is a series of the form $\sum_{n=0}^{\infty} C_n x^n$ where C_n are the coefficients of the series
- Example: $C_n = 1$ for all n gives the geometric series
- We can generalize the power series to $\sum_{n=0}^{\infty} C_n (x-a)^n$, which is a power series *about* a
  - Note $x^0 = (x-a)^0 = 1$ even when $x = 0$ or $x = a$
  - Therefore $x = a \implies \sum_{n=0}^{\infty} C_n (x-a)^n = C_0$ which always converges
- Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \cdot \frac{n^2}{(n+1)^2} \underset{n\to\infty}{\to} |x|$
  - Therefore this series converges absolutely when $|x| < 1$ and diverges $|x| > 1$

- Special case when $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$ which converges
- When $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the alternating series test
- Therefore this series converges for $-1 \leq x \leq 1$

- Example: $\sum_{n=0}^{\infty} \frac{(1+5^n)x^n}{n!}$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(1+5^{n+1})x^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+5^n)x^n}\right| = \frac{1+5^{n+1}}{1+5^n}\left|\frac{x}{n+1}\right| \underset{n\to\infty}{\to} 0$
  - Therefore this series converges absolutely for all value of x
- Example: $\sum_{n=0}^{\infty} n!x^n$
  - Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|x| \underset{n\to\infty}{\to} \infty$
  - This diverges for all value of x, *except* $x = 0$
- Theorem: For a power series of the form $\sum_{n=0}^{\infty} C_n(x-a)^n$ has 3 possibilities:
  1. The series converges only when $x = a$
  2. The series converges for all x
  3. The series converges for $|x - a| < R$
     - R is known as the *radius of convergence*
     - The *interval of convergence* is the interval of x for which the series is convergent; this may or may not include end points
- Note the power series will always converge for $x = a$ no matter what the series is
- Typically the ratio test is used to determine the radius of convergence, but will not work for the end points (which are often considered as special cases)

Lecture 17, Feb 18, 2022

Representing Functions as Power Series

- Example: $\frac{x}{x-3}$
  - $\frac{x}{x-3} = -x \cdot \frac{1}{3-x} = -\frac{x}{3} \cdot \frac{1}{1-\frac{x}{3}} = -\frac{x}{3}\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right) = -\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n+1}$
  - Converges for $|x| < 3$
- Theorem: Term-by-Term Differentiation and Integration: For $\sum C_n(x-a)^n$ with radius of convergence $R = R_0 > 0$ then $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ is continuous and differentiable on $(a - R_0, a + R_0)$, and:
  - $f'(x) = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$
    * Note the sum now starts from $n = 1$, because there is a constant term that disappears
    * Alternatively, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} C_n(x-a)^n\right) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(C_n(x-a)^n\right)$
  - $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1}$
    * Alternatively, $\int \left(\sum_{n=0}^{\infty} C_n(x-a)^n\right)\mathrm{d}x = \sum_{n=0}^{\infty} \int \left(C_n(x-a)^n\right)\mathrm{d}x$
  - Both derived series have the same radius of convergence, but the behaviour at end points may

change

- This allows us to calculate some otherwise difficult integrals, e.g. $\displaystyle\int_0^{0.1} \frac{1}{1+x^4}\,\mathrm{d}x$
- Example: $\dfrac{1}{(1+x)^2}$
  - Note $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{1}{1+x}\right) = \dfrac{1}{(1+x)^2}$
  - $\dfrac{1}{(1+x)^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{1}{1+x}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\displaystyle\sum_{n=0}^{\infty}(-x)^n\right) = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} = \displaystyle\sum_{n=1}^{\infty}(-1)^n(n+1)x^n$
- Example: $\ln(1-x)$
  - $\ln(1-x) = -\displaystyle\int \frac{1}{1-x}\,\mathrm{d}x = -\int \sum_{n=0}^{\infty} x^n\,\mathrm{d}x = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -\sum_{n=0}^{\infty}\frac{x^n}{n}$
  - Solve for the constant of integration: Set $x = 0$, $\ln(1-0) = 0 = C$
- Example: $\tan^{-1} x$
  - $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \dfrac{1}{1+x^2}$
  - $\tan^{-1} x = \displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}\,\mathrm{d}x = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$
  - Calculate constant of integration by $x = 0$: $C = \tan^{-1} 0 = 0$
  - $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
  - Radius of convergence $R = 1$ follows from the original geometric series; we can test the boundaries to see it also converges for $x = \pm 1$
  - This leads to $\tan^{-1} = \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$ (Leibniz formula for π)

Taylor and Maclaurin Series

- Let $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$ for $|x-a| < R$
  - Notice that $f(a) = C_0$
  - $f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots \implies f'(a) = C_1$
  - $f''(x) = 2C_2 + 6C_3(x-a) + 12C_4(x-a)^2 + \cdots \implies f''(a) = 2C_2$
  - Following this pattern we note that $f^{(n)}(a) = n!C_n$
- Theorem: If $f(x)$ has a power series representation about a of $f(x) = \displaystyle\sum_{n=0}^{\infty} C_n(x-a)^n$ for $|x-a| < R$ then the coefficients are given by $C_n = \dfrac{f^{(n)}(a)}{n!}$
  - $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(x-a)^n = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots$
  - This is known as the *Taylor series* of f about a
  - In the special case where $a = 0 \implies f(x) = f(0) + f'(0)x + \dfrac{1}{2!}f''(0)x^2 + \cdots$ this is called a Maclaurin series
- Definition: A function is *analytic* at a if it can be represented as a power series about a
  - The function essentially needs to be infinitely differentiable at a
- Example: $f(x) = e^x$ about 0
  - All derivatives at 0 are equal to 1
  - $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$
  - To determine our ratio of convergence use the ratio test: $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \dfrac{x}{n+1} \underset{n\to\infty}{\to} 0$ so the series converges for all x

Lecture 18, Feb 28, 2022

Taylor's Theorem

- When is a function actually equal to its Taylor expansion?
  - Define a partial sum for the Taylor series: $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ (the n-th degree Taylor polynomial of f about a)
  - Define the remainder as $R_n(x) = f(x) - T_n(x)$
- Theorem: If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ then f is equal to its Taylor series expansion
- Taylor's Theorem: Given $f', f'', \cdots, f^{(n+1)}$ exists and are continuous on an open interval I, and $a \in I$, then $\forall x \in I, f(x) = T_n(x) + R_n(x)$ where $R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t$
  - Proof:
    * Consider: $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$ by FTC
    * Integration by parts: $\int_a^b f'(t)\,\mathrm{d}t = [f'(t)(t-b)]_a^b - \int_a^b f''(x)(t-b)\,\mathrm{d}t$

$$\begin{aligned}
&= f'(a)(b-a) + \int_a^b f''(t)(b-t)\,\mathrm{d}t \\
&= f'(a)(b-a) + \left[-\frac{(b-t)^2}{2}f''(t)\right]_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \\
&= f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2!} + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \\
&= \cdots
\end{aligned}$$

    * Applying this n times: $\int_a^b f'(t)\,\mathrm{d}t = \sum_{i=1}^{n} \frac{(b-a)^i}{i!}f^{(i)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t = f(b) - f(a)$
    * Let $x = b \implies f(x) = \sum_{i=0}^{n} \frac{(x-a)^i}{i!}f^{(i)}(a) + R_n(x)$ where $R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t$
- To prove that a function is equal to its Taylor series we need to prove $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(x)\,\mathrm{d}t = 0$; the integral form is not always the most convenient to work with
  - If we can bound the derivative: For $|f^{(x+1)}(t)| \leq M$ for $a < t < x$: $|R_n(x)| \leq$

$$\begin{aligned}
&\left|\int_a^x \frac{M(x-t)^n}{n!}\,\mathrm{d}t\right| \\
&= \left|M\left[\frac{(x-t)^{n+1}}{(n+1)!}\right]_a^x\right| \\
&= M\frac{|x-a|^{n+1}}{(n+1)!}
\end{aligned}$$

  - Or using the MVT, $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ for $c \in (a, x)$
- Example: Prove e^x is equal to the sum of its Taylor series
  - $f^{(n+1)}(t) = e^t$
  - For a Taylor series about 0 the range is $0 < t < x \implies e^t < e^x = M$, or $x < t < 0 \implies e^t < 1 = M$
  - $R_n(x) \leq \frac{e^x x^{n+1}}{(n+1)!} \underset{n\to\infty}{\to} 0$ (by sequence limit 4)
  - Since the remainder goes to 0 the Taylor series converges to e^x for all x

Taylor Series Examples

- Example: Maclaurin series for $\cos x$
  - $f(x) = \cos x \implies f(0) = 1$
  - $f'(x) = -\sin x \implies f'(0) = 0$
  - $f''(x) = -\cos x \implies f''(0) = -1$
  - $f'''(x) = \sin x \implies f'''(x) = 0$
  - $f''''(x) = \cos x$ so the cycle repeats
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ because all the odd terms are zero
  - Use ratio test to determine radius of convergence (all x)
  - Note for all derivatives the magnitude is always ≤ 1: $|R_n(x)| \leq \left|\frac{x^{n+1}}{(n+1)!}\right| \underset{n\to\infty}{\to} 0$ so the Taylor series sum to $\cos x$
- As long as the derivative doesn't tend to infinity, R_n always goes to 0
- Since the coefficients of a Taylor series are unique we can obtain them in other methods; e.g. differentiating the series of $\cos x$ to get $\sin x$ or multiplying by x to get $x \sin x$
- Example: Taylor series for $\cos x$ about $\frac{17\pi}{4}$
  - This series is useful despite the series for $\cos x$ converging for all x due to rate of convergence
  - Derivatives:
    * $f(x) = \cos x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * $f'(x) = -\sin x \implies f\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
    * $f''(x) = -\cos x \implies f\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
    * $f'''(x) = \sin x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * $f''''(x) = \cos x \implies f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}}$
    * There are two negatives and two positives alternating so we need to use 2 sums
  - $\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{17\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{\left(x - \frac{17\pi}{4}\right)^2}{2!} + \cdots$
  - $\cos x = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(-1)^n \frac{\left(x - \frac{17\pi}{4}\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(-1)^{n+1} \frac{\left(x - \frac{17\pi}{4}\right)^{2n+1}}{(2n+1)!}$
- Example: Prove $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ for $x \in (-1, 1]$
  - Derivatives:
    * $f(x) = \ln(1+x)$
    * $f'(x) = \frac{1}{1+x}$
    * $f''(x) = \frac{-1}{(1+x)^2}$
    * $f'''(x) = \frac{2}{(1+x)^3}$
    * $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$
  - We can't bound this derivative in $-1 < x \leq 1$ because as x tends to -1 the derivative shoots off to infinity, so we need to work with the integral form

- $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t$
$= \frac{1}{n!}\int_0^x (-1)^{n+1}\frac{n!}{(1+t)^{n+1}}(x-t)^n \,\mathrm{d}t$
$= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \,\mathrm{d}t$
- For $0 \leq x \leq 1$: $|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t \leq \int_0^x (x-t)^n \,\mathrm{d}t = \frac{x^{n+1}}{n+1} \underset{n\to\infty}{\to} 0$ since $x < 1$
- For $-1 < x < 0$: $|R_n(x)| = \left|\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t\right|$
$= \int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t}\,\mathrm{d}t$
  * Apply MVT: $\int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t}\,\mathrm{d}t = \left(\frac{z-x}{1+z}\right)^n \frac{1}{1+z}(-x)$, where $x < z < 0$ and $-x$ is the interval width $b-a$
  * To show $\frac{z-x}{1+z} < 1$:
$$\begin{aligned} & |x| < 1 \\ \implies & |x| - |z| < 1 - |z| \\ \implies & \frac{|x|-|z|}{1-|z|} < 1 \\ \implies & \frac{-x+z}{1+z} < 1 \\ \implies & \frac{z-x}{1+z} < 1 \end{aligned}$$
  * $\lim_{n\to\infty}\left(\frac{z-x}{1+z}\right)^n = 0 \implies R_n(x) = \left(\frac{z-x}{1+z}\right)^n \frac{1}{1+z}(-x) \underset{n\to\infty}{\to} 0$

Lecture 19, Mar 1, 2022

Multiplication and Division of Power Series

- Example: $\frac{e^x}{1-x}$
  - We can find its Taylor expansion by multiplying two series together: $\left(1 + x + \frac{x^2}{2!} + \cdots\right)(1 + x + x^2 + \cdots)$
- Generally the radius of convergence of a product or ratio is the smaller of the two radii
- Example: $\tan x = \frac{\sin x}{\cos x}$
  - $\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots}$
  - Note here the radius of convergence is $|x| < \frac{\pi}{2}$

Applications of Taylor Polynomials

- Error bounds can be found for Taylor approximations:
  1. If it's an alternating series, then like all alternating series the error bound is just the next term
  2. Otherwise, the error bound can be computed using Taylor's theorem $R_n = \int_a^x \frac{(t-a)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t < M\frac{(x-a)^{n+1}}{(n+1)!}$
- Example: Using the Taylor expansion of $\sqrt{x}$ about $a = 1$, evaluate $\sqrt{1.25}$
  - Derivatives:

    - $f(x) = x^{\frac{1}{2}} \implies f(1) = 1$
    - $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \implies f'(1) = \frac{1}{2}$
    - $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \implies f''(1) = -\frac{1}{4}$
    - $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} \implies f'''(1) = \frac{3}{8}$
    - $f''''(x) = -\frac{15}{16}x^{-\frac{7}{2}} \implies f''''(1) = \frac{-15}{16}$
  - $\sqrt{x} \approx T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!} + \frac{3}{8}\frac{(x-1)^3}{3!}$
    - Since this is an alternating series: $|R_3(x)| < |a_4| = \frac{15}{16}\frac{(x-1)^4}{4!}$
  - $\sqrt{1.25} \approx 1.11816 \pm 0.00015$
- Example: Maximum error for the Maclaurin series of $\cos x$ for $|x| < \frac{\pi}{4}$ for $n = 3$
  - $\cos x \approx T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
  - Another alternating series, so the uncertainty is $|R_3(x)| < \frac{x^8}{8!} < \frac{\left(\frac{\pi}{4}\right)^8}{8!} = 3.6 \times 10^{-6}$
  - Alternatively, using the Taylor remainder formula $|R_3(x)| < 1\left|\frac{x^8}{8!}\right|$
- Example: Find $\ln(1.4)$ to within 0.001 with $\ln(1-x)$
  - $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$
  - When x is negative this is an alternating series so we can use the next term as an error bound
  - $|R_5(x)| = \left|\frac{x^6}{6}\right| = \frac{0.4^6}{6} = 0.0007$ so we need to take 5 terms

Lecture 20, Mar 4, 2022

The Binomial Series

- Binomial theorem: $(a+b)^k = \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n$ where $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} = \frac{k!}{(k-n)!n!}$ and $\binom{k}{0} = 1$
- Consider $(1+x)^k = \sum_{n=0}^{k}\binom{k}{n}x^n$ which is a power series
- With the binomial theorem, we assumed that k is a positive integer; now we show that this works for all values of k but the series becomes infinite
- Using the Maclaurin series for $(1+x)^k$:
  - Derivatives:
    - $f(x) = (1+x)^k \implies f(0) = 1$
    - $f'(x) = k(1+x)^{k-1} \implies f'(0) = k$
    - $f''(x) = k(k-1)(1+x)^{k-2} \implies f''(0) = k(k-1)$
    - $f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$
  - $f(x) = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n = \sum_{n=0}^{\infty}\binom{k}{n}x^n$
  - Notice that if k is a positive integer, then at $n = k+1$ we get a zero term in the derivative, which truncates the series
- Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k-n}{n+1}x\right| \underset{n\to\infty}{\to} |x|$
  - The binomial series converges for $|x| < 1$
  - For the endpoints:

        * $k \leq -1 \implies I = (-1, 1)$
        * $-1 < k < 0 \implies I = (-1, 1]$
        * $k \geq 0 \implies I = [-1, 1]$
- Example: $f(x) = \dfrac{1}{\sqrt{2+x}} = (2+x)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2}}\left(1 + \dfrac{x}{2}\right)^{-\frac{1}{2}} \implies k = -\dfrac{1}{2}$
    - $f(x) = \dfrac{1}{\sqrt{2}}\displaystyle\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\frac{x}{2} + \frac{3}{4}\frac{\left(\frac{x}{2}\right)^2}{2!} + \cdots\right)$
    - Radius of convergence: $\left|\dfrac{x}{2}\right| < 1 \implies |x| < 2, R = 2$
    - Since $-1 < k = -\dfrac{1}{2} < 0$ the interval of convergence is $(-2, 2]$
- Linearization: If we take the first degree Taylor polynomial we get $(1+x)^k \approx 1 + kx$

Fourier Series

- Fourier series allow us to represent any periodic function as an infinite sum of sines and cosines
- Definition: A function is periodic with period T if $f(t+T) = f(t)$ for all t
    - The smallest T for which this holds is called the *fundamental period*
    - e.g. $\cos(\pi t)$ has a fundamental period of 2; $\sin(2\pi t)$ has a fundamental period of 1; $\cos(\pi t) + \sin(2\pi t)$ has a fundamental period of 2, the larger of the 2 periods
- Theorem: Let $f(t)$ be a piecewise continuous and piecewise differentiable periodic function with fundamental period T, then $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n\cos(n\omega t) + b_n\sin(n\omega t)$ where $\omega = \dfrac{2\pi}{T}$ (known as the *Fourier* series of f)
    - Note: Piecewise differentiability means we only have a finite number of places where the function is not differentiable
    - a_0 is divided by 2 to allow us to write a single definition for the a_ns
    - a_n, b_n are known as the Fourier coefficients; finding a Fourier series is all about finding these coefficients
- Note $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & n \neq 0 \\ T & n = 0 \end{cases}$ as the positive and negative parts exactly cancel each other out
- Similarly $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\sin(n\omega t)\,\mathrm{d}t = 0$
- $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}}\sin(m\omega t)\sin(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ \dfrac{T}{2} & m = n \end{cases}$
- $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos(m\omega t)\sin(n\omega t)\,\mathrm{d}t = 0$
    - See section 7.2
- To find the Fourier coefficients, we multiply the series through by sine or cosine and then integrate, so that all terms except for the one we want go to zero
    - $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(m\omega t)\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} a_m\cos(m\omega t)\cos(m\omega t)\,\mathrm{d}t + \sum\int_{-\frac{T}{2}}^{\frac{T}{2}} a_n\cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t = \frac{T}{2}a_m$
- Fourier coefficients are given by $a_n = \dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega t)\,\mathrm{d}t, b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega t)\,\mathrm{d}t$
    - $n = 0, 1, \cdots$ for a_n but starts at 1 for b_n
        * For $n = 0$, b_0 is just 0 so we skip it
        * Notice that for $n = 0$, $a_0 = \dfrac{2}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t$ which is just the average value of f over one period (times 2)
- Example: Triangle function with height of π and period of 2π, with peak at the origin

  - $T = 2\pi, \omega = \dfrac{2\pi}{T} = 1$
  - The function can be represented by $f(t) = \pi - |t|, t \in [-\pi, \pi]$
  - $b_n = \dfrac{2}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin(n\omega t)\,\mathrm{d}t$

$$\begin{aligned}
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |t|)\sin(nt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{\pi}\pi\sin(nt)\,\mathrm{d}t - \int_{-\pi}^{0}(-t)\sin(nt)\,\mathrm{d}t - \int_{0}^{\pi}t\sin(nt)\,\mathrm{d}t\right) \\
&= -\frac{1}{\pi}\left(-\frac{1}{n^2}\left[\sin(nt) - nt\cos(nt)\right]_{-\pi}^{0} + \frac{1}{n^2}\left[\sin(nt) - nt\cos(nt)\right]_{0}^{\pi}\right) \\
&= 0
\end{aligned}$$

    * This makes sense because the triangle function is an even function, but the b_n terms correspond to sines which are odd, so they shouldn't have any contribution
  - $a_n = \dfrac{2}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos(n\omega t)\,\mathrm{d}t$

$$\begin{aligned}
&= 2\frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\cos(nt)\,\mathrm{d}t \\
&= \begin{cases} \pi & n = 0 \\ 0 & n \neq 0 \text{ and odd} \\ \dfrac{4}{\pi n^2} & n \neq 0 \text{ and even} \end{cases}
\end{aligned}$$

  - $f(t) = \dfrac{\pi}{2} + \displaystyle\sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)^2}\cos((2k-1)t)$
  - Note $f(0) = \pi = \dfrac{\pi}{2} + \displaystyle\sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)^2} \implies \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$
- Example: $f(t) = t^2, t \in [-\pi, \pi]$
  - $T = 2\pi, \omega = \dfrac{2\pi}{T} = 1$
  - This is an even function so $b_n = 0$
  - $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} t^2\,\mathrm{d}t = \frac{2\pi^2}{3}$
  - $a_n = \dfrac{2}{2\pi}\displaystyle\int_{-\pi}^{\pi} t^2\cos(nt)\,\mathrm{d}t = \begin{cases} -\dfrac{4}{n^2} & n \text{ odd} \\ \dfrac{4}{n^2} & n \text{ even} \end{cases}$
  - $f(t) = \dfrac{\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nt)$

Lecture 21, Mar 7, 2022

Vectors, Lines and Planes

- $\vec{a} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$
- In general vectors don't have a particular location in space where it starts (they represent displacement instead of a location)
- Radius vectors denoted $\vec{r}$ start at $(0, 0, 0)$
- A plane is described by $ax + by + cz = d$ or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
  - (a, b, c) is the normal vector of the plane, (x_0, y_0, z_0) is a point on the plane
- Planes are defined uniquely by a point on the plane and a normal vector
  - Let $\vec{n} = (n_1, n_2, n_3)$ be the normal vector and $\vec{r}_0 = (x_0, y_0, z_0)$ be the point on the plane
  - Let $p = (x, y, z)$ be on the plane connected to the origin by $\vec{r}$

  - We constrain the plane by noting that any $\vec{r} - \vec{r}_0$ is in the plane and thus normal to $\vec{n}$, therefore $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$
    * $n(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$
- A line is described by $\vec{r} = \vec{r_0} + t\vec{v} \implies (x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$ parametrically or $\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$ (the 2-equation symmetric form)
- Lines are defined uniquely by a point on the line and a parallel vector
  - Let $\vec{v}$ be the direction vector and $\vec{r}_0$ be the point on the line
  - Any other point $\vec{r}$ on the line must have $\vec{r} - \vec{r}_0 \propto \vec{v} \implies \vec{r} = \vec{r}_0 + t\vec{v}$
- Parametric equations in 3D are often written as position vectors $\vec{r}(t)$

Lecture 22, Mar 8, 2022

Surfaces: Cylinders and Quadric Surfaces

- Quadric surfaces are in the form $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$
  - $Gx + Hy + Iz$ offsets the shape in the axial directions
  - $Dxy + Exz + Fyz$ terms rotate the shape
- Cylinders are 2D surfaces with no z defined
- Properties of quadric surfaces:
  1. Domain/range
     - Typically domain involves x and y, range involves z, but not always
  2. Intercepts with coordinate axes (all 3)
  3. Traces – intersections with coordinate planes
  4. Sections – intersections with other planes (typically a plane parallel to the coordinate planes)
  5. Centre
  6. Symmetry
  7. Bounded/unboundedness
- Example: Hyperboloid of two sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \implies z = \pm c\sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}$
  1. Domain: $x, y \in (-\infty, \infty)$; range: $z \geq c$ or $z \leq -c$ (since the number in the root is always greater than 1)
  2. Intercepts: z intercept at $z = \pm c$ (when $x = y = 0$) (we can't have $z = 0$ so it can't have other intercepts)
  3. Traces:
     - xy plane ($z = 0$): nothing
     - xz plane ($y = 0$): hyperbola $z = \pm c\sqrt{1 + \frac{x^2}{a^2}}$
     - yz plane ($x = 0$): hyperbola $z = \pm c\sqrt{1 + \frac{y^2}{b^2}}$
  4. Sections: consider $z = z_0$ where $|z_0| > c$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0^2}{c^2} - 1$ where $\frac{z_0^2}{c^2} - 1 > 0$, which is an ellipse
  5. Centre: origin because there are no offset terms
  6. Symmetry: $x \to -x$, $y \to -y$, $z \to -z$ all give the same surface so symmetry about each axis
  7. Unbounded in all directions

Projections

- For 2 intersecting surfaces, we will have a curve of intersection $C : (x, y, z)$ such that $z = f(x, y) = g(x, y)$
  - The relation $f(x, y) = g(x, y)$ is a vertical cylinder since there are no restrictions on z
- Fix $z = 0$ and have $f(x, y) = g(x, y) = 0$; this is a *projection* (like a shadow of the curve of intersection)
- Example: cone $x^2 + y^2 = 2z^2 \implies z = \pm\sqrt{\frac{x^2 + y^2}{2}}$ and a plane $y + 4z = 5 \implies z = \frac{5 - y}{4}$

- Set $$\sqrt{\frac{x^2+y^2}{2}} = \frac{5-y}{4}$$
$$\implies (5-y)^2 = 8(x^2+y^2)$$
$$\implies 25 - 10y + y^2 = 8x^2 + 8y^2$$
$$\implies 8x^2 + 7y^2 + 10y - 25 = 0$$
$$\implies \frac{x^2}{\frac{25}{7}} + \frac{\left(y+\frac{5}{7}\right)^2}{\frac{200}{49}} = 1$$
- This is an offset ellipse

Vector Functions and Limits

- A vector valued function or vector function is $\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k} = (f_1(t), f_2(t), f_3(t))$
  - Like a parametric representation
- Definition: $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L}$ if $\lim\limits_{t\to t_0} \left\|\vec{f}(t) - \vec{L}\right\| = 0$
  - The norm turns the vector limit into an ordinary limit
- $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L} \implies \lim\limits_{t\to t_0} \left\|\vec{f}(t)\right\| = \left\|\vec{L}\right\|$ but the reverse is not true
- Limit rules: Given $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L}$, $\lim\limits_{t\to t_0} \vec{g}(t) = \vec{M}$, $\lim\limits_{t\to t_0} u(t) = R$:
  1. $\lim\limits_{t\to t_0} \vec{f}(t) + \vec{g}(t) = \vec{L} + \vec{M}$
  2. $\lim\limits_{t\to t_0} \alpha\vec{f}(t) = \alpha\vec{L}$
  3. $\lim\limits_{t\to t_0} u(t)\vec{f}(t) = R\vec{L}$
  4. $\lim\limits_{t\to t_0} \vec{f}(t)\cdot\vec{g}(t) = \vec{L}\cdot\vec{M}$
  5. $\lim\limits_{t\to t_0} \vec{f}(t)\times\vec{g}(t) = \vec{L}\times\vec{M}$
- $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L}$ iff $\lim\limits_{t\to t_0} f_1(t) = L_1$, $\lim\limits_{t\to t_0} f_2(t) = L_2$, $\lim\limits_{t\to t_0} f_3(t) = L_3$
- $\vec{f}(t)$ is continuous at t_0 if $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{f}(t_0)$
- Likewise the continuity of $\vec{f}$depends on the continuity of its component

Lecture 23, Mar 11, 2022

Vector Derivatives and Integrals

- The derivative of a vector function is defined as $\vec{f}'(t) \equiv \lim\limits_{h\to 0} \dfrac{\vec{f}(t+h) - \vec{f}(t)}{h}$
- Derivative can be taken componentwise: $\vec{f}'(t) = f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k}$
  - Proof: $$\begin{aligned}\vec{f}'(t) &= \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \\ &= \lim_{h\to 0}\left[\frac{f_1(t+h)-f_1(t)}{h}\hat{\imath} + \frac{f_2(t+h)-f_2(t)}{h}\hat{\jmath} + \frac{f_3(t+h)-f_3(t)}{h}\hat{k}\right] \\ &= \lim_{h\to 0}\frac{f_1(t+h)-f_1(t)}{h}\hat{\imath} + \lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h}\hat{\jmath} + \lim_{h\to 0}\frac{f_3(t+h)-f_3(t)}{h}\hat{k} \\ &= f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k}\end{aligned}$$
- Integrals can also be defined componentwise as $\displaystyle\int_a^b \vec{f}(t)\,\mathrm{d}t = \left(\int_a^b f_1(t)\,\mathrm{d}t\right)\hat{\imath} + \left(\int_a^b f_2(t)\,\mathrm{d}t\right)\hat{\jmath} + \left(\int_a^b f_3(t)\,\mathrm{d}t\right)\hat{k}$

- All ordinary derivative and integral properties apply:
  - $\int_a^b \vec{c}\cdot\vec{f}(t)\,\mathrm{d}t = \vec{c}\cdot\int_a^b \vec{f}(t)\,\mathrm{d}t$
  - $\left\|\int_a^b \vec{f}(t)\,\mathrm{d}t\right\| \leq \int_a^b \left\|\vec{f}(t)\right\|\mathrm{d}t$

Differentiation Formulas

- Define a composition function $(\vec{f}\circ u)(t) = \vec{f}(u(t))$
  - Note this composition can't go the other way around, because $\vec{f}$ takes in a scalar and u takes in a vector, so $u(\vec{f}(t))$ makes no sense
- Differentiation rules:
  1. $(\vec{f}+\vec{g})'(t) = \vec{f}'(t)+\vec{g}'(t)$
  2. $(\alpha\vec{f})'(t) = \alpha\vec{f}'(t)$
  3. $(u\vec{f})'(t) = u(t)\vec{f}'(t) + u'(t)\vec{f}(t)$
  4. $(\vec{f}\cdot\vec{g})'(t) = \vec{f}(t)\cdot\vec{g}'(t) + \vec{f}'(t)\cdot\vec{g}(t)$
  5. $(\vec{f}\times\vec{g})'(t) = \vec{f}(t)\times\vec{g}'(t) + \vec{f}'(t)\times\vec{g}(t)$
     - Note that for this one, order matters since cross product is non-commutative
  6. $(\vec{f}\circ u)'(t) = u'(t)\vec{f}(u(t))$
- Example: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
  - Define $r \equiv \|\vec{r}\| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{x^2+y^2+z^2} \implies \vec{r}\cdot\vec{r} = r^2$
  - $\vec{r}\cdot\vec{r} = r^2 \implies \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\cdot\vec{r} + \vec{r}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = 2r\frac{\mathrm{d}r}{\mathrm{d}t} \implies \vec{r}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = r\frac{\mathrm{d}r}{\mathrm{d}t}$
- Example: $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r}}{r}$
  - This is a unit vector in the direction of $\vec{r}$; even though the magnitude is constant, the derivative can be nonzero since the direction can change
  - $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{r}}{r} &= \frac{1}{r}\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\vec{r} \\ &= \frac{1}{r^3}\left(r^2\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - r\frac{\mathrm{d}r}{\mathrm{d}t}\vec{r}\right) \\ &= \frac{1}{r^3}\left((\vec{r}\cdot\vec{r})\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} - \left(\vec{r}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)\vec{r}\right) \\ &= \frac{1}{r^3}\left(\left(\vec{r}\times\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)\times\vec{r}\right)\end{aligned}$$
    - Note we used the relationship $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$

Curves

- $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$
- The derivative $\vec{r}'(t)$ is interpreted geometrically as a vector pointing in the tangent direction of the curve
- Definition: Let C be parameterized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ and be differentiable; then $\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$ (if not $\vec{0}$) is tangent to C at $(x(t), y(t), z(t))$ and $\vec{r}'(t)$ points in the direction of increasing t
- Example: Find tangent to $\vec{r}(t) = (1+2t)\hat{i} + t^3\hat{j} + \frac{t}{2}\hat{k}$ at $(9, 64, 2)$
  - First find the t value: $\vec{r}(4) = (9, 64, 2)$
  - $\vec{r}'(t) = 2\hat{i} + 3t^3\hat{j} + \frac{1}{2}\hat{k} \implies \vec{r}'(4) = 2\hat{i} + 48\hat{j} + \frac{1}{2}\hat{k}$
  - The tangent line is $\vec{R}(q) = 9\hat{i} + 64\hat{j} + 2\hat{k} + q\left(2\hat{i} + 48\hat{j} + \frac{1}{2}\hat{k}\right)$

- Define the *unit tangent vector* as $\vec{T}(t) \equiv \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$
  - Note $\vec{T}(t) \cdot \vec{T}(t) = 1$ since $\vec{T}$ is a unit vector
  - Differentiating this leads to $\vec{T}'(t) \cdot \vec{T}(t) = 0$
  - $\vec{T}'(t)$ is always in the perpendicular direction to $\vec{T}$; this is because $\vec{T}$ has a constant magnitude so the derivative can only represent a change in direction, which is always perpendicular
  - $\vec{T}'(t)$ is telling you the direction that the curve is curving, similar to how the second derivative tells you whether the function is concave up or down

Arc Length

- Extended to 3D, arc length is $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,\mathrm{d}t = \int_a^b \|\vec{r}'(t)\|\,\mathrm{d}t$
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{r}'(t) = 3\cos(t)\hat{\imath} - 3\sin(t)\hat{\jmath} + 4\hat{k}$
  - $\|\vec{r}'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$
  - $\displaystyle\int_0^2 \pi\|\vec{r}'(t)\|\,\mathrm{d}t = 10\pi$
- Sometimes a curve is parameterized with respect to arc length instead of t
- Example: $\vec{r}(t) = t^2\hat{\imath} + t^2\hat{\jmath} - t^2\hat{k}$ from $(0, 0, 0)$
  - $$\begin{aligned} s &= \int_0^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau \\ &= \int_0^t \sqrt{4\tau^2 + 4\tau^2 + 4\tau^2}\,\mathrm{d}\tau \\ &= \int_0^3 2\sqrt{3}\tau\,\mathrm{d}\tau \\ &= \sqrt{3}t^2 \end{aligned}$$
  - $s = \sqrt{3}t^2 \implies t^2 = \dfrac{s}{\sqrt{3}} \implies \vec{r}(s) = \dfrac{s}{\sqrt{3}}\hat{\imath} + \dfrac{s}{\sqrt{3}}\hat{\jmath} - \dfrac{s}{\sqrt{3}}\hat{k}$

Lecture 24, Mar 14, 2022

Curvature in 2D

- Definition: Curvature is defined as $\kappa \equiv \left|\dfrac{\mathrm{d}\phi}{\mathrm{d}s}\right|$, where in 2D ϕ is the angle that the tangent line makes with the x axis and s is arc length
  - Let $y = y(x) \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = y' = \tan\phi \implies \phi = \tan^{-1} y' \implies \dfrac{\mathrm{d}\phi}{\mathrm{d}x} = \dfrac{y''}{1 + (y')^2}$
  - Note $\dfrac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (y'(x))^2}$
  - $\dfrac{\mathrm{d}\phi}{\mathrm{d}x} = \dfrac{\mathrm{d}\phi}{\mathrm{d}s}\dfrac{\mathrm{d}s}{\mathrm{d}x} = \dfrac{\mathrm{d}\phi}{\mathrm{d}s}\sqrt{1 + (y')^2}$
  - $\dfrac{\mathrm{d}\phi}{\mathrm{d}s}\sqrt{1 + (y')^2} = \dfrac{y''}{1 + (y')^2} \implies \dfrac{y''}{(1 + (y')^2)^{\frac{3}{2}}}$
  - Therefore $\kappa = \dfrac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$ (all derivatives are with respect to x)

- For a parametric curve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y'}{x'} \implies \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{x'y'' - y'x''}{(x')^3} \implies \kappa = \dfrac{\left|\frac{x'y''-y'x''}{(x')^3}\right|}{\left(1+\left(\frac{y'}{x'}\right)^2\right)^{\frac{3}{2}}} = \dfrac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}}$

- Example: circle $\vec{r} = r\cos(t)\hat{\imath} + r\sin(t)\hat{\jmath}$
  - $x(t) = r\cos t \implies x'(t) = -r\sin t \implies r'' = -r\cos t$
  - $y(t) = r\sin t \implies y' = r\cos t \implies y'' = -r\sin t$
  - $\kappa = \dfrac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}} = \dfrac{|r^2\sin^2 t + r^2\cos^2 t|}{(r^2\sin^2 t + r^2\cos^2 t)^{\frac{3}{2}}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$
  - We can also get this intuitively by noting that $\dfrac{\Delta\phi}{\Delta s} = \dfrac{2\pi}{2\pi r} = \dfrac{1}{r}$
- This leads to the definition for the *radius of curvature*: $\rho = \dfrac{1}{\kappa}$
  - The radius of curvature is the radius of the tangent circle to the curve at any given point
- Note: Consider the unit tangent vector $\vec{T} = \cos(\phi)\hat{\imath} + \sin(\phi)\hat{\jmath}$
  - $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s} = -\sin(\phi)\dfrac{\mathrm{d}\phi}{\mathrm{d}s}\hat{\imath} + \cos(\phi)\dfrac{\mathrm{d}\phi}{\mathrm{d}s}\hat{\jmath}$
  - $\left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\| = \left|\dfrac{\mathrm{d}\phi}{\mathrm{d}s}\right| = \kappa$
  - This gives an alternative interpretation of curvature as the rate of change of the unit tangent vector with respect to arc length

Curvature in 3D

- Definition: $\kappa \equiv \left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\|$ in 3D, where $\vec{T}$ is the unit tangent vector to the curve and s is the arc length
- Consider $C : \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$
  - $\kappa = \left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}\dfrac{\mathrm{d}t}{\mathrm{d}s}\right\| = \dfrac{\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|}{\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|} = \dfrac{\left\|\vec{T}'\right\|}{\|\vec{r}'\|}$
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{r}'(t) = 3\cos(t)\hat{\imath} - 3\sin(t)\hat{\jmath} + 4\hat{k}$
  - $\|\vec{r}'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$
  - $\vec{T} = \dfrac{\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}}{\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\|} = \dfrac{3}{5}\cos(t)\hat{\imath} - \dfrac{3}{5}\sin(t)\hat{\jmath} + \dfrac{4}{5}\hat{k}$
  - $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = -\dfrac{3}{5}\sin(t)\hat{\imath} - \dfrac{3}{5}\cos(t)\hat{\jmath} + 0\hat{k} \implies \left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| = \dfrac{3}{5}$
  - $\kappa = \dfrac{\frac{3}{5}}{5} = \dfrac{3}{25}$ or $\rho = \dfrac{25}{3}$
- Alternatively: $\kappa = \dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

The Normal and Binormal Vectors

- Definition: The *principal unit normal* $\vec{N}(t) \equiv \dfrac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$, i.e. a unit vector in the direction of $\vec{T}'$
  - $\vec{T}$ is tangent to the curve and $\vec{N}$ is perpendicular to this tangent
  - $\vec{N}$ points in the direction that the curve is changing
- Definition: The *osculating plane* is the plane defined by $\vec{N}$ and $\vec{T}$
  - The osculating plane is the plane that comes closest to containing the curve at a point
- Definition: The *binormal vector* is $\vec{B} = \vec{T} \times \vec{N}$
- Example: for a straight line $\vec{T}'(t) = 0$ since the tangent vector doesn't change; this means $\vec{N}$ does not exist and we can't define an osculating plane
  - This can be interpreted as a straight line is contained in an infinite number of planes
- Example: Circular helix $\vec{r}(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$ for $t \in [0, 2\pi]$
  - $\vec{T}'(t) = -\dfrac{3}{5}\sin(t)\hat{\imath} - \dfrac{3}{5}\cos(t)\hat{\jmath} \implies \left\|\vec{T}'(t)\right\| = \dfrac{3}{5}$
  - $\vec{N}(t) = -\sin(t)\hat{\imath} - \cos(t)\hat{\jmath}$
  - $\vec{B} = \left(\dfrac{4}{5}\cos t, -\dfrac{4}{5}\sin t, -\dfrac{3}{5}\right)$
  - A point on the plane is $(3\sin t, 3\cos t, 4t)$
  - The equation of the plane is $\left(\dfrac{4}{5}\cos t\right)(x - 3\sin t) - \left(\dfrac{4}{5}\sin t\right)(y - 3\cos t) - \dfrac{3}{5}(z - 4t) = 0 \implies 4\cos(t)x - 4\sin(t)y - 4z = -12t$
- In general the magnitude of the binormal vector is always 1, because $\left\|\vec{B}\right\| = \left\|\vec{T}\right\|\left\|\vec{N}\right\|\sin\theta = 1$
  - $\vec{T}, \vec{N}, \vec{B}$ form a set of mutually perpendicular unit vectors
  - We can use this as a new coordinate system, useful in some physical situations e.g. satellites

Lecture 25, Mar 15, 2022

Motion in Space: Velocity and Acceleration

- Interpret $\vec{r}(t)$ as the location, then $\vec{r}'(t) = \vec{v}(t)$ is the velocity and $\vec{r}''(t) = \vec{a}(t)$ is the acceleration and $\dfrac{\mathrm{d}s}{\mathrm{d}t} = \|\vec{r}'(t)\|$ is the speed
- For circular motion around the origin $\vec{r}(t) = a\cos(\theta(t))\hat{\imath} + a\sin(\theta(t))\hat{\jmath}$
  - θ' is the rate of change of the angle, so $\theta' > 0$ is a counterclockwise movement
  - Define θ' as the *angular velocity* in radians per second; $|\theta'|$ is the *angular speed*
    * Note θ' is essentially a scalar property but with a sign
  - If $\theta' = \omega$ then
$$\begin{aligned} & \theta = \omega t \\ \implies & \vec{r} = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath} \\ \implies & \vec{v} = -r\omega\sin(\omega t)\hat{\imath} + r\omega\cos(\omega t)\hat{\jmath} \\ \implies & \vec{a} = -r\omega^2\cos(\omega t)\hat{\imath} - r\omega^2\sin(\omega t)\hat{\jmath} = -\omega^2\vec{r} \end{aligned}$$

Vector Mechanics

- Newton's second law: $\vec{F}(t) = m\vec{a}(t)$
- Momentum: $\vec{p}(t) = m\vec{r}'(t)$ and $\vec{p}'(t) = \vec{F}(t)$
- Conservation of momentum: $\vec{p}' = \vec{F} = 0 \implies \vec{p}$ is constant
- Angular momentum: $\vec{L} \equiv \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ where $\vec{r}$ is the radius vector
  - $\left\|\vec{L}\right\| = m\|\vec{r}\|\|\vec{v}\|\sin(\theta(t))$ where θ is the angle between $\vec{v}$ and $\vec{r}$
  - We can break up $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$ (directions perpendicular and parallel to $\vec{r}$)
  - $\vec{L} = m\vec{r} \times \vec{v} = m\vec{r} \times (\vec{v}_\perp + \vec{v}_\parallel) = m\vec{r} \times \vec{v}_\perp$ since $\vec{r} \times \vec{v}_\parallel = 0$

- Angular momentum only cares about the component of $\vec{v}$ perpendicular to $\vec{r}$
- Example: $\vec{r}(t) = r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath}$
  - $\vec{L} = m\vec{r} \times \vec{v} = m(0, 0, r^2\omega\cos^2(\omega t) + r^2\omega\sin^2(\omega t)) = (0, 0, mr^2\omega)$
- Example: Uniform motion in a straight line
  - $\vec{r} = \vec{r}_0 + t\vec{v}$
  - $\vec{L} = m\vec{r} \times \vec{v} = m(\vec{r}_0 + t\vec{v}) \times \vec{v} = m\vec{r}_0 \times \vec{v}$ which is a constant assuming $\vec{v}$ is constant
  - Uniform motion in a straight line has a constant angular momentum
- Torque: $\vec{L}' = m\vec{r} \times \vec{r}'' + m\vec{r}' \times \vec{r}' = \vec{r} \times \vec{F} \equiv \vec{\tau}$
- Central force: Definition: $\vec{F}$ is a *central* or *radial force* if $\vec{F}(t)$ is always parallel to $\vec{r}$
  - Example: gravity, electric field associated with a point charge
  - $\vec{r} \times \vec{F} = 0 \implies \tau = 0 \implies \vec{L}$ is constant
  - Angular momentum is conserved when only a central force is present

Acceleration

- Break apart acceleration $\vec{a} = \vec{a}_n + \vec{a}_t$ (normal and tangential directions)
- $\vec{T} = \dfrac{\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}}{\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\|} = \dfrac{\vec{v}}{\frac{\mathrm{d}s}{\mathrm{d}t}} \implies \vec{v} = \dfrac{\mathrm{d}s}{\mathrm{d}t}\vec{T}$
  - Differentiating, we get $\vec{v}' = \vec{a} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{T} + \dfrac{\mathrm{d}s}{\mathrm{d}t}\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}$
  - Note $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \left\|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|\vec{N}$ and $\kappa = \dfrac{\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|}{\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|}$
  - $\vec{a} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}\vec{T} + \kappa\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right)^2\vec{N}$
    - The first term is $\vec{a}_t$, the tangential component, the second is $\vec{a}_n$, the normal component
- $\|\vec{a}_t\| = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ which is the rate of change of speed in the direction of motion
  - Example: uniform circular motion with $\dfrac{\mathrm{d}s}{\mathrm{d}t}$ equal to a constant so $\|\vec{a}_t\| = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$ so there is no tangential acceleration
- $\|\vec{a}_n\| = \kappa\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right)^2$ directed perpendicular to the tangent, in the direction of curvature
  - Alternatively $\|\vec{a}_n\| = \dfrac{\|\vec{v}\|^2}{\rho}$ where $\rho = \dfrac{1}{\kappa}$ is the radius of curvature

Lecture 26, Mar 18, 2022

Initial Value Problems

- Using relationships like Newton's second law and relationship between angular momentum and torque to calculate the motion of objects
- Example: Confinement of a hot plasma by a magnetic field for fusion energy
  - Mean speed in a thermal distribution is about $v_{th} = \sqrt{\dfrac{kT}{m}}$ where $k = 1.4 \times 10^{-23}$ is the Boltzmann constant and temperature is on the order of 10×10^8 K, $m_{ion} = 1.6 \times 10^{-27}\,\mathrm{kg}, m_e = 9.1 \times 10^{-31}\,\mathrm{kg}$
    - At this temperature hydrogen nuclei are moving at about 1×10^6 m/s while electrons move at 3×10^7 m/s and start experiencing relativistic effects
    - We need to constrain the nuclei so it doesn't just fly off
      - To do this, a magnetic bottle is used (Tokamak)
- Charged particles in a magnetic field experiences $\vec{F} = q\vec{E}$ from the electric field and $\vec{F} = q\vec{v} \times \vec{B}$ from the magnetic field (Lorentz force)
  - Note that the magnetic field only has an impact when the particle is moving, and that the direction of the magnetic force is perpendicular to both the velocity and the field

- Assume we have no electric field and orient the constant magnetic field along the z axis, so that $\vec{B} = B_0\hat{k}$
  * Strength of the magnetic field for fusion is about 10T over a large volume, which requires superconducting coils
- $\vec{F} = m\vec{v}' = qB_0\vec{v} \times \vec{k} \implies \vec{v}' = \omega_L \vec{v} \times \hat{k}$ where $\omega_L \equiv \dfrac{qB_0}{m}$
- Let $\vec{v}(t) = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath} + v_2(t)\hat{k}$
- $v_x'\hat{\imath} + v_y'\hat{\jmath} + v_z'\hat{k} = \omega_L v_y \hat{\imath} - \omega_L v_x \hat{\jmath} \implies \begin{cases} v_x'(t) = \omega_L v_y(t) \\ v_y'(t) = -\omega_L v_x(t) \\ v_z'(t) = 0 \end{cases}$
  * We have a system of coupled differential equations
- In the z direction, v_z is a constant since $v_z' = 0$
  * A particle moving in the same direction of the magnetic field is unaffected by the field
- In the x direction, $v_x'' = \omega_L v_y' = -\omega_L^2 v_x$, so the solution is $v_x(t) = A\sin(\omega_L t + \phi)$
- In the y direction, $v_y = \dfrac{1}{w_L} v_x' \implies v_y(t) = A\cos(\omega_L t + \phi)$
- Thus $\vec{v}(t) = A\sin(\omega_L t + \phi)\hat{\imath} + A\cos(\omega_L t + \phi)\hat{\jmath} + c\hat{k}$
- $\vec{r}(t) = \left(-\dfrac{A}{\omega_L}\cos(\omega_L t + \phi) + D_x\right)\hat{\imath} + \left(\dfrac{A}{\omega_L}\sin(\omega_L + \phi) + D_y\right)\hat{\jmath} + (ct + D_z)\,\hat{k}$
  * Set $\phi = D_x = D_y = D_z = 0 \implies \vec{r}(t) = -r_L\cos(\omega_L t)\hat{\imath} + r_L\sin(\omega_L t + \hat{\jmath}) + ct\hat{k}$ where $r_L = \dfrac{A}{\omega_L}$
  * This is a circular helix
- ω_L is known as the gyro frequency or Larmor frequency; for electrons about 1×10^{13} Hz (microwaves) for electrons or 6×10^9 Hz for ions (radio waves)
  * This gives us a mechanism for heating up the plasma as ω_L is a natural resonant frequency for all the particles

- The particle is unconstrained in the direction of the magnetic field, which is problematic – we try to get around this by making the magnetic field a torus
  - However, the magnetic field is no longer uniform, and there are now electric fields
- Consider constant magnetic field pointing out of the page and constant electric field pointing up and a stationary particle
  - Initially, the magnetic field has no affect, but the electric field still moves the particle
  - The particle starts moving in the direction of the electric field (up), but now the magnetic field has an effect and causes it to turn right
  - As the particle turns, it starts to move against the electric field and lose energy
  - Eventually the particle ends up in the same vertical position as where it started, as it loses all kinetic energy to the electric field, and the cycle repeats
- Let $\vec{B} = (0, 0, B_z), \vec{E} = (0, E_y, 0)$
  - $v_x'(t) = \omega_L v_y(t)$ and $v_z'(t) = 0$ remains unchanged
  - $v_y'(t) = -\omega_L v_x(t) + \dfrac{qE_y}{m}$ with an additional electric force component
  - Uncoupling these and solving we get $\begin{cases} v_x = v_\perp \sin(\omega t) + \dfrac{E_y}{B_z} \\ v_y = v_\perp \cos(\omega t) \\ v_t = c \end{cases}$
    * The additional $v_D = \dfrac{E_y}{B_z}$ is known as the *drift velocity*, causing a constant motion in the x direction
    * In the general case, $\vec{v}_D = \dfrac{\vec{E} \times \vec{B}}{B^2}$
    * The drift velocity has no dependence on charge or mass; all particles in the plasma move at the same speed

Lecture 27, Mar 21, 2022

Multivariable Functions

- The domain of a multivariable function is an entire region
- One way to visualize a multivariable function is to use a *level map* or *contour plot* (i.e. sections)
  - Cut through the surface with a plane parallel to the xy plane, i.e. set $f(x, y) = c$
  - This gives us a curve and we can plot several of these contours for different c on the same plot
  - This is referred to as a *collection of level curves*
- Example: $f(x, y) = xy$
  - Set $f = c \implies xy = c \implies y = \dfrac{c}{x}$
  - The level curves are a collection of $\dfrac{c}{x}$; for positive c the curve is in the first and third quadrant, for negative c the curve is in the second and fourth quadrant
  - The curve looks like a saddle increasing along positive xy and decreasing along negative xy
- For a function of 3 variables, we can extend this idea to make *level surfaces*, so we can get and idea of what the function looks like without a fourth dimension

Limits of Multivariable Functions

- Notation: $f(x, y, z) = f(\vec{x})$
- There are an infinite number of ways to approach a point in multiple dimensions
- Definition: $\lim\limits_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L \iff \forall\varepsilon > 0, \exists\delta > 0 \ni 0 < \|\vec{x} - \vec{x}_0\| < \delta \implies |f(\vec{x}) - L| < \varepsilon$
  - δ defines a circle, sphere, etc around the point $\vec{x}_0$
  - In the single variable case we only had left and right hand limits; with multiple variables there are an infinite number of ways
- Example: $\lim\limits_{x\to\vec{0}} \dfrac{x^2y + y^2}{x + y^2}$
  - Try approaching the origin from the positive y axis:
    * $x = 0 \implies f(0, y) = \dfrac{y^2}{y^2} = 1 \implies \lim\limits_{y\to 0^+} f(0, y) = 1$
  - Approaching from the x axis:
    * $y = 0 \implies f(x, 0) = \dfrac{0}{x} = 0 \implies \lim\limits_{x\to 0^+} f(x, 0) = 0$
  - This limit does not exist because we get different values for it when we approach from different directions
- Example: $\lim\limits_{\vec{x}\to\vec{0}} \dfrac{x^2y^4}{x^4 + y^8}$
  - Consider the path $y = mx$:
    * $f(x, mx) = \dfrac{m^4x^6}{x^4 + m^8x^8} = \dfrac{m^4x^2}{1 + m^8x^4} \implies \lim\limits_{x\to 0} f(x, mx) = 0$
  - Consider a parabolic path $x = y^2$:
    * $f(y^2, y) = \dfrac{y^4y^4}{y^8 + y^8} = \dfrac{y^8}{2y^8} = \dfrac{1}{2} \implies \lim\limits_{y\to 0} f(y^2, y) = \dfrac{1}{2}$
  - Again the limit does not exist
- To prove that the limit actually does exist we need an epsilon delta proof
- Example: $\lim\limits_{\vec{x}\to\vec{0}} \dfrac{2xy^2}{x^2 + y^2}$
  - Impose $\varepsilon > 0$, require $|f - L| = \left|\dfrac{2xy^2}{x^2 + y^2} - 0\right| = \dfrac{2y^2|x|}{x^2 + y^2} < \varepsilon$, when $0 < \|\vec{x} - \vec{x}_0\| = \sqrt{x^2 + y^2} < \delta$

- $$y^2 \leq x^2 + y^2$$
  $$\implies \frac{y^2}{y^2} \geq \frac{y^2}{x^2+y^2}$$
  $$\implies \frac{2y^2|x|}{x^2+y^2} \leq \frac{2y^2|x|}{y^2} = 2|x|$$
  $$\implies 2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2+y^2} < 2\delta$$
  * Therefore choose $\delta = \frac{\varepsilon}{2} \implies \left|\frac{2y^2x}{x^2+y^2}\right| < \varepsilon$
- Therefore $\lim_{\vec{x}\to\vec{0}} \frac{2xy^2}{x^2+y^2} = 0$

Lecture 28, Mar 22, 2022

Continuity of Multivariable Functions

- Definition: A multivariable function f is continuous at $\vec{x}_0$ if $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$
  - Example: $f(x,y) = \frac{x^2+3xy}{x^2-2y}$ is continuous everywhere except $x^2 - 2y = 0$, i.e. the curve $y = \frac{1}{2}x^2$ is excluded
- Theorem: Continuity of composite functions: If g is continuous at $\vec{x}_0$ and f is continuous at $g(\vec{x}_0)$ then $f(g(\vec{x}))$ is continuous at $\vec{x}_0$
  - As in a vector valued function case, the composition can only go in one direction
- If $f(\vec{x})$ is continuous at $\vec{x}_0$, then it is continuous in both variables; however continuity in both variables does not imply continuity as a whole
  - $f(\vec{x})$ is continuous at $\vec{x}_0 \implies \lim_{x\to x_0} f(x, y_0) = f(x_0, y_0)$ and $\lim_{y\to y_0} f(x_0, y) = f(x_0, y_0)$
  - Example: Consider $f(x, mx) = \frac{m^4x^6}{x^4 + m^8x^8}$ from last lecture; the limit exists if approaching from any direction in a straight line so it works for the x and y directions, but any other path yields a different limit
  - Example: $f(x,y) = \begin{cases} \frac{xy^2}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous along both axes, but along the line $y = 3x \implies f(x, 3x) = \frac{9x^3}{28x^3} = \frac{9}{28}$ so the function is not continuous

Partial Derivatives

- Partial derivatives treat all variables except one as a constant and differentiate with respect to that variable
- Definition: Partial derivative of f with respect to x is $f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$
  - Similarly $f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$
- Interpret as slicing the graph with a plane, and then taking the slope of the tangent line of the curve on that plane
- Since partial derivatives are also functions of multiple variables, higher order partial derivatives can be taken, with respect to the same variable or different variables
  - We can mix partial derivatives, e.g. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
  - Note in the notation $\frac{\partial^2 f}{\partial y \partial x}$, the derivative is first taken with respect to x and then y, i.e. denominator is read right-to-left; in the notation f_{xy}, the subscript is read left-to-right, i.e. derivative is taken with respect to x and then y

- Clairaut's Theorem (symmetry of second partial derivatives): $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ on every open set for which $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are continuous
  - This applies to 3 variables as well for $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$, etc

Partial Differential Equations

- When derivatives of more than one variable occur in a relation, it is a partial differential equation
- Example: Laplace's Equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (fluid flow)
- Example: 1D wave equation: $\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$

Lecture 29, Mar 25, 2022

Redefining the Derivative

- A partial derivative only gives the rate of change along one of the axes; to define differentiability for a multivariable function, we need to consider all directions
- If we try to simply extend the definition of a derivative as $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x})}{\vec{h}}$ we run into the problem of dividing by a vector; if we change it to $\left\|\vec{h}\right\|$ instead, we lose information about the direction; thus we need to reinvent the derivative
- Definition: Little o notation: $g(h) = o(h)$ if $\lim_{h \to 0} \frac{g(h)}{|h|} = 0$, i.e. $g(h)$ goes to 0 faster than h goes to 0
  - Big O is used to indicate that two things are the same order of magnitude; little o is used for different orders of magnitude
- In one dimension,
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ \implies \lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} &= 0 \\ \implies (f(x+h) - f(x)) - f'(x)h &= o(h) \end{aligned}$$
  - Example: $f(x) = x^2$
    - $f(x+h) - f(x) = (x+h)^2 - x^2 = (2x)h + h^2$
    - By our definition $2x$ is f' if h^2 is $o(h)$
    - $\lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \implies h^2 = o(h)$

Differentiability in Multiple Dimensions

- Extending our definition of little o to multiple variables: $\lim_{\vec{h} \to \vec{0}} \frac{g(\vec{h})}{\left\|\vec{h}\right\|} = 0 \implies g(\vec{h}) = o(\vec{h})$
- Definition: f is differentiable at $\vec{x} \iff \exists \vec{y} \ni f(\vec{x} + \vec{h}) - f(\vec{x}) = \vec{y} \cdot \vec{h} + o(\vec{h})$
  - $\vec{y} = \nabla f(\vec{x})$ is the *gradient* of $\vec{f}$
- Example: $f(x, y) = x + y^2$
  - Let $\vec{h} = (h_1, h_2)$
  - $$\begin{aligned} f(\vec{x} + \vec{h}) - f(\vec{x}) &= f(x + h_1, y + h_2) - f(x, y) \\ &= x + h_1 + (y + h_2)^2 - x - y^2 \\ &= h_1 + 2yh_2 + h_2^2 \\ &= (1\hat{\imath} + 2y\hat{\jmath}) \cdot \vec{h} + h_2^2 \end{aligned}$$

- Let $g(\vec{h}) = h_2^2 = h_2\hat{j} \cdot \vec{h} \implies \frac{|g(\vec{h})|}{\left\|\vec{h}\right\|} = \frac{\left\|h_2\hat{j}\right\|\left\|\vec{h}\right\|\cos\theta}{\left\|\vec{h}\right\|} \leq |h_2| \implies \lim_{\vec{h}\to\vec{0}} \frac{|g(\vec{h})|}{\left\|\vec{h}\right\|} = 0$
  - Since g is $o(\vec{h})$ the gradient is $1\hat{i} + 2y\hat{j}$
- Theorem: $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\mathrm{d}f}{\mathrm{d}y}\hat{j} + \frac{\mathrm{d}f}{\mathrm{d}z}\hat{k}$
  - Note $f(\vec{x})$ is a vector but $\nabla f(\vec{x})$ is a vector
  - If the gradient exists, then the function is differentiable at that point
- The gradient points in the direction of steepest ascent

Directional Derivatives

- The idea of a partial derivative can be extended beyond just the axes
- Definition: The directional derivative of f at $\vec{x}_0$ in the direction of $\hat{u}$ is $f_{\hat{u}}(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\hat{u}) - f(\vec{x}_0)}{h}$
  - Note $\hat{u}$ is a unit vector by definition
- The direction derivative is related to the gradient: $f_{\hat{u}}(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \hat{u}$
  - Proof: $f(\vec{x} + \vec{h}) - f(\vec{x}) = \Delta f(\vec{x})\vec{h} + o(\vec{h})$
    - Choose $\vec{h} = h\hat{u}$ where $h = \left\|\vec{h}\right\| \implies f(\vec{x} + h\hat{u}) - f(\vec{x}) = \nabla f(\vec{x}) \cdot h\hat{u} + o(\vec{h}) \implies \lim_{h\to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \lim_{h\to 0} \nabla f(\vec{x}) \cdot \hat{u} + \frac{o(\vec{h})}{h} \implies f_{\hat{u}} = \nabla f(\vec{x}_0) \cdot \hat{u}$
- Example: Parabolic hill $z(x, y) = 20 - x^2 - y^2$
- Note: $|f_{\hat{u}}(\vec{x})| = |\nabla f \cdot \hat{u}| = \|\nabla f\|\|\hat{u}\||\cos\theta| \leq \|\nabla f\|$
  - The rate of change along any direction is always less than or equal to the rate of change along the gradient
  - Max rate of change happens for $\theta = 0$, i.e. $\hat{u}$ pointing in the direction of ∇f
  - This shows that the gradient points in the direction with the greatest rate of change
- Example: Project the path of steepest descent to the xy plane: $z = f(x, y) = A + x + 2y - x^2 - 3y^2$ from $(0, 0, A)$
  - $\nabla f = (1 - 2x)\hat{i} + (2 - 6y)\hat{j} \implies -\nabla f = (2x - 1)\hat{i} + (6y - 2)\hat{j}$
  - Consider the curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$; we always want it to point in the direction of steepest descent so $x'(t), y'(t)$ should be in the opposite direction of the gradient
  - $\begin{cases} x'(t) = 2x(t) - 1 \\ y'(t) = 6y(t) - 2 \end{cases} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6y - 2}{2x - 1}$
  - This is a separable differential equation; solving gives $6y - 2 = (2x - 1)^3 e^C$; plugging in the initial point we get $e^C = 2$
  - $y = \frac{(2x - 1)^3}{3} = \frac{1}{3}$

Lecture 30, Mar 28, 2022

The Chain Rule

- Theorem: Multivariable chain rule: $\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$
  - Recall that $\vec{r}' = \vec{T}\frac{\mathrm{d}x}{\mathrm{d}t}$ so $\nabla f(\vec{r}) \cdot \vec{r}' = \nabla f \cdot \vec{T}\frac{\mathrm{d}s}{\mathrm{d}t}$
    - $\nabla f\vec{T}$ is the directional derivative in the direction of $\vec{T}$
    - Multiplying the rate of change in a direction by the speed gives the overall rate of change
- Example: Rectangular volume $V = lhd$, let $\frac{\mathrm{d}l}{\mathrm{d}t} = 3\text{m/s}$, $\frac{\mathrm{d}h}{\mathrm{d}t} = -2\text{m/s}$, $\frac{\mathrm{d}d}{\mathrm{d}t} = 5\text{m/s}$, what is $\frac{\mathrm{d}V}{\mathrm{d}t}$ at $l = 2\text{m}, h = 3\text{m}, d = 4\text{m}$?
  - Let $\vec{q}(t) = (l, h, d)$

- $\frac{\mathrm{d}V}{\mathrm{d}t} = \nabla V(\vec{q}(t)) \cdot \vec{q}'(t) = (hd, ld, hl) \cdot (3, -2, 5) = 3hd - 2ld + 5lh = 50\mathrm{m}^3 s$
    - A "position vector" can be something other than just spacial directions
- We can extend this further and make $x(t, s), y(t, s)$ also functions of multiple variables; then $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$ etc (note now we have all partial derivatives)

Revisiting Implicit Differentiation

- How do we get $\frac{\mathrm{d}y}{\mathrm{d}x}$ from an implicit relation $u(x, y) = 0$?
    - Let $x = t, y = y(t)$
- $u = u(t, y(t)) \implies \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$
    - Since $u(t, y(t)) = 0$ we have $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$
    - $x = t \implies \frac{\mathrm{d}x}{\mathrm{d}t} = 1$
    - $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}$
    - $0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$
- Example: $x^4 + 4x^3y + y^4 = 1$
    - Let $u = x^4 + 4x^3y + y^4 - 1 = 0$
    - $\frac{\partial u}{\partial x} = 4x^3 + 12x^2y$
    - $\frac{\partial u}{\partial y} = 4x^3 + 4y^3$
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2(x + 3y)}{x^3 + y^3}$
- Same result can be obtained by doing implicit differentiation normally, but this method can be easier

Level Curves in 2D

- The gradient is always normal to the level curve (i.e. perpendicular to the tangent of the normal curve)
    - $f(x, y) = c$ is the level curve; let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ describe this curve, then $\vec{t} = \vec{r}'(t)$
    - $f(\vec{r}(t)) = c$ must hold for this to be the level curve
    - $\frac{\mathrm{d}}{\mathrm{d}t}f(\vec{r}(t)) = \nabla f(\vec{r}) \cdot \vec{r}' = \frac{\mathrm{d}}{\mathrm{d}t}c = 0$
    - Therefore $\nabla f \perp \vec{r}'$ or $\nabla f \perp \vec{t}$
- This works for any curve in the form $f(x, y) = c$
- Using this we can obtain the expression for $\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$
    - $\nabla f \cdot \vec{t} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} = 0$
- To get the equation of the tangent line: $(x - x_0, y - y_0) \cdot \nabla f = 0 \implies (x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} = 0$
- Similarly for the normal line: $(x - x_0, y - y_0) \cdot \vec{t} = 0 \implies (x - x_0)\frac{\partial f}{\partial y} - (y - y_0)\frac{\partial f}{\partial x} = 0$
- Example: $x^2 + y^2 = 9$
    - $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y \implies (x - x_0)2x_0 + (y - y_0)2y_0 = 0$
    - Choose e.g. the point $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

- Tangent is $\left(x - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} + \left(y - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} = 0 \implies y = \frac{6}{\sqrt{2}} - x$

Lecture 31, Mar 29, 2022

Functions of 3 Variables: Level Surfaces

- Level surface: $f(x, y, z) = c$
- By extension from the 2D case, the gradient is perpendicular to the level surface (tangent plane to the surface)
  - This can be shown in the same way as in the 2D case
  - Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be any curve on the surface
  - $f(x(t), y(t), z(t)) = f(\vec{r}(t)) = c \implies \frac{\mathrm{d}}{\mathrm{d}t}f(\vec{r}(t)) = 0 \implies \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = 0$
- Equation for the tangent plane is given by $(\vec{x} - \vec{x}_0) \cdot \nabla f(\vec{x}_0) = (x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y} + (z - z_0)\frac{\partial f}{\partial z} = 0$
- Normal line equation is given by $\vec{r}(q) = \vec{x}_0 + q\nabla f(\vec{x}_0)$
- Example: $xy^2 + 2z^2 = 12$, find normal line at $(1, 2, 2)$
  - $\begin{cases} \frac{\partial f}{\partial x} = y^2 = 4 \\ \frac{\partial f}{\partial y} = 2xy = 4 \\ \frac{\partial f}{\partial z} = 4z = 8 \end{cases} \implies \begin{cases} x = 1 + 4q \\ y = 2 + 4q \\ z = 2 + 8q \end{cases}$
- Example: offset sphere $f = x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$ and ellipsoid $g = x^2 + 3y^2 + 2z^2 = 9$, show that the sphere is tangent to the ellipsoid at $(2, 1, 1)$
  - Show that gradient vectors are parallel
  - $\nabla f = (2x - 8)\hat{i} + (2y - 8)\hat{j} + (2z - 6)\hat{k} \implies \nabla f(2, 1, 1) = (-4, -6, -4)$
  - $\nabla g = 2x\hat{i} + 6y\hat{j} + 4z\hat{k} \implies \nabla g(2, 1, 1) = (4, 6, 4) = -\nabla f(2, 1, 1)$
  - Note we also need to show that the point is on both surfaces

Lecture 32, Apr 1, 2022

Multivariable Optimization

- Definition: f has a local maximum at $\vec{x}_0$ iff $f(\vec{x}_0) \geq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$; f has a local minimum iff $f(\vec{x}_0) \leq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$
- Theorem: If f has a local extreme value at $\vec{x}_0$ then $\nabla f(\vec{x}_0) = \vec{0}$ or $\nabla f(\vec{x}_0)$ does not exist
  - Proof:
    * Let $g(x) = f(x, y_0)$, a single variable function; this function must have an extreme value at $\vec{x}_0$ if f has an extreme value at $\vec{x}_0$
    * $\frac{\mathrm{d}g}{\mathrm{d}x}(x_0) = 0 = \frac{\partial f}{\partial x}(x_0, y_0)$
  - The tangent plane is horizontal when this happens: Consider the level surface $z = f(x, y)$, let $g(x, y, z) = z - f(x, y) = 0$, then $\nabla g = \hat{k}$ for $f_x = f_y = 0$ so the normal is pointing straight up, which means the tangent plane is horizontal
  - However, this *doesn't* mean that the gradient equalling zero or DNE implies the existence of a local extreme
- Definition: Points where $\nabla f = \vec{0}$ or DNE are *critical points*; where $\nabla f = \vec{0}$ are stationary points; stationary points that are not extrema are *saddle points*
- Example: $f(x, y) = 2x^2 + y^2 - xy - 7y$
  - $\nabla f = (4x - y, 2y - x - 7) = \vec{0} \implies \begin{cases} x = 1 \\ y = 4 \end{cases}$
  - $f(1, 4) = -14$

- Look in the neighbourhood of $(1,4)$: in all directions f is a little larger, so this point is a local minimum
- Note the max/min could be an entire curve rather than a single point (e.g. a torus)
- Theorem: Second Derivatives Test: For $f(x,y)$ with continuous second order partials and $\nabla f(\vec{x}_0) = \vec{0}$, set $\begin{cases} A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \\ B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{cases}$ and form the discriminant $D = AC - B^2$, then:
  1. If $D < 0$, then (x_0, y_0) is a saddle point
  2. If $D > 0$ and $A, C > 0$, then (x_0, y_0) is a local minimum
     - Note this is only possible if A and C have the same sign
  3. If $D > 0$ and $A, C < 0$ then (x_0, y_0) is a local maximum
  4. In all other cases, the result is indeterminate
- The second derivative test in 2D is a special case of the more general second derivative test, which looks at the Hessian matrix; if the Hessian is positive definite, then the point is a local minimum; if the Hessian is negative definite, it is a local maximum; if the eigenvalues are mixed positive and negative, then it is a saddle point; in all other cases (zero eigenvalues), it is indeterminate

Lecture 33, Apr 4, 2022

Absolute Extrema

- As with single variable optimization, the absolute min/max can occur at a local min/max or on the boundary (which are curves in the multivariable case)
- Theorem: If f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set (extreme value in multiple variables)
- First find critical points inside the region where the gradient is zero or DNE, and then find critical points along the boundary, and then the end points
  - By parameterizing the boundary curve, we can find its derivative and find when that equals zero
  - Either use the chain rule or substitute back into the original function
- Example: $f(x,y) = (x-4)^2 + y^2$ on $\left\{ (x,y) : 0 \leq x \leq 2, x^3 \leq y \leq 4x \right\}$
  - Critical points: $\nabla f = (2(x-4), 2y) = \vec{0} \implies x = 4, y = 0$
    * The critical point is not in the set we're looking at, which means both absolute max and min are on the boundary
  - Boundary 1: $y = x^3, 0 \leq x \leq 2$
    * Parameterize the boundary $\begin{cases} x = t \\ y = t^3 \end{cases}, 0 \leq t \leq 2 \implies \vec{r}_1(t) = t\hat{i} + t^3\hat{j}$
    * Now we have a single variable function
    * $\frac{\mathrm{d}}{\mathrm{d}t} f_1(\vec{r}_1(t)) = \nabla f_1 \cdot \vec{r}_1'(t) = (2(t-4), 2t^3) \cdot (1, 3t^2) = 2t - 8 + 6t^5$
    * Let $f_1'(t) = 0 \implies t(1 + 3t^4) = 4 \implies t = 1$ or the point $(1,1)$, which is in our set
    * $f(1,1) = 10$
    * $f_1''(t) = 2 + 30t^4 = 32 > 0$ so this is a minimum
  - Boundary 2: $y = 4x, 0 \leq x \leq 2$
    * Parameterize the boundary $\begin{cases} x = t \\ y = 4t \end{cases}$
    * $f_2(t) = (t-4)^2 + (4t)^2 = 17t - 8t + 16 \implies f_2'(t) = 34t - 8 = 0 \implies t = \frac{4}{17}$
    * Critical point is at $\left(\frac{4}{17}, \frac{16}{17}\right)$
  - Test for whether these are minimums or maximums with a second derivative test

  - Check endpoints $f(0,0) = 16, f(2,8) = 68$
- Example: $f(x,y) = xy^2 - x$ on $\{\, (x,y) \mid x^2 + y^2 \leq 3 \,\}$
  - $\nabla f = (y^2 - 1)\hat{\imath} + 2xy\hat{\jmath} = \vec{0} \implies \begin{cases} y^2 - 1 = 0 \\ 2xy = 0 \end{cases} \implies \begin{cases} y = \pm 1 \\ x = 0 \end{cases}$
  - Using second derivative test, both are saddle points
  - Check boundary $y^2 = 3 - x^2 \implies f_1(x) = x(3 - x^2) - x = 2x - x^3 \implies f_1'(x) = 2 - 3x^2 \implies x = \pm\sqrt{\frac{3}{2}}, y = \pm\sqrt{\frac{7}{3}}$
  - There are 4 critical points on the boundary; using the second derivative test we can determine which one is max or min
  - Even though the circle has no end points, when we expressed it as $y^2 = 3 - x^2$ we introduced constraints of $-\sqrt{3} \leq x \leq \sqrt{3}$, so we must treat those as end points of the boundary
  - Another approach is to parameterize the boundary curve as $\vec{r}(t) = \sqrt{3}\cos t\hat{\imath} + \sqrt{3}\sin t\hat{\jmath}, 0 \leq t \leq 2\pi$, and then taking the derivative of $f(\vec{r}(t))$, which removes the need for end points

Lecture 34, Apr 5, 2022

Lagrange Multipliers

- In general the goal is to maximize or minimize $f(x,y)$ subject to a constraint of $g(x,y) = k$
  - Geometrically, picture the level curves of $f(x,y)$ along with the curve $g(x,y) = k$
  - A solution must lie on both $g(x,y) = k$ and one of the level curves of $f(x,y)$; the goal is to find the largest c such that $f(x,y) = c$ intersects $g(x,y) = k$
  - This happens when the two curves just touch each other at a single point, i.e. they're tangent to each other
    * Note if they crossed, there would always be a way to choose a larger or smaller c such that they touch at a single point
- Since the curves are tangent, they share the same tangent and thus $\nabla g \parallel \nabla f$ or $\nabla f = \lambda \nabla g$; λ is the *Lagrange multiplier*
- In the 2D case, we need to solve $\begin{cases} g(x_0, y_0) = k \\ f_x(x_0, y_0) = \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0) \end{cases}$
  - Together we have 3 equations and 3 unknowns x_0, y_0, λ, so we can solve the system
  - Often we don't care about finding λ, only x_0, y_0
- In 3D, the surfaces are tangent and share the same tangent plane so again the gradients are parallel
  - In general $\begin{cases} g(\vec{x}_0) = k \\ \nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0) \end{cases}$
- Example: $f(x,y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$
  - $\nabla f = (2x, -2y), \nabla g = (2x, 2y)$
  - $\begin{cases} x_0^2 + y_0^2 = 1 \\ 2x_0 = \lambda 2x_0 \\ -2y_0 = \lambda 2y_0 \end{cases}$
  - From the second equation, either $x_0 = 0$ or $\lambda = 1$; from the third equation either $y_0 = 0$ or $\lambda = -1$
  - Cases:
    1. $\lambda = 1, y_0 = 0 \implies x_0 = \pm 1$
    2. $\lambda = -1, x_0 = 0 \implies y_0 = \pm 1$
- Note Lagrange's method doesn't tell us whether we have a max or min, but it does give us all the max/min
- Example: Maximize $f(x,y,z) = xyz$ subject to $x^3 + y^3 + z^3 = 1, x, y, z \geq 0$

- $\begin{cases} x^3+y^3+z^3=1 \\ yz=\lambda 3x^2 \\ xz=\lambda 3y^2 \\ zy=\lambda 3z^2 \end{cases} \implies \begin{cases} xyz=\lambda 3x^2 \\ xyz=\lambda 3y^2 \\ xyz=\lambda 3z^2 \end{cases}$
  - $\lambda x^3=\lambda y^3=\lambda z^3$
  - Eliminate $\lambda=0$ possibility because if $\lambda=0$, $x=y=z=0$ which would be a minimum
  - $x^3=y^3=z^3 \implies x=y=z$
  - Plugging this back into our constraint we get $x=y=z=\sqrt[3]{\frac{1}{3}} \implies f(x,y,z)=\frac{1}{3}$
- Problems of this type are easy to set up, but solving the system of equations is complicated

Two Constraints Problem

- Maximize or minimize $f(x,y,z)$ subject to $g(x,y,z)=k$ and $h(x,y,z)=c$
  - Geometrically we're trying to maximize or minimize f on the intersection between g and h
  - The 3D surfaces $g(x,y,z)=k$ and $h(x,y,z)=c$ intersect at a curve
- Note that since the gradient is normal to a level surface, $\vec{T}$ for the intersection curve is normal to both ∇h and ∇g
  - Therefore $\vec{T}=\nabla h\times\nabla g$
- By the same logic as before, $\vec{T}$ must be in the tangent plane of f at the max/min, so ∇f is perpendicular to $\vec{T}$
  - Since $\nabla f\perp\vec{T}$, it must be in the plane defined by ∇g and ∇h since that plane is also perpendicular to $\vec{T}$
  - $\nabla f(\vec{x}_0)=\lambda\nabla g(\vec{x}_0)+\mu\nabla h(\vec{x}_0)$ is our new equation
- The problem is now reduced to a set of 5 equations
- Example: $f(x,y,z)=xy+2z$ on a circle of intersection between the plane $x+y+z=0$ and the sphere $x^2+y^2+z^2=24$
  - $\nabla f=(y,x,2), \nabla g=(1,1,1), \nabla h=(2x,2y,2z)$
  - $\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=24 \\ y=\lambda+\mu\cdot 2x \\ x=\lambda+\mu\cdot 2y \\ 2=\lambda+\mu\cdot 2z \end{cases}$
  - $(x-y)=2\mu(y-x) \implies (x-y)(1+2\mu)=0 \implies y=x$ or $\mu=-\frac{1}{2}$
  - Cases:
    1. $x=y \implies 2x+z=0 \implies z=-2x \implies x^2+x^2+(-2x)^2=24 \implies 6x^2=24 \implies x=\pm 2, y=\pm 2, z=\mp 4$
       * This produces the points $f(2,2,-4)=-4$ and $f(-2,-2,4)=12$
    2. $\mu=-\frac{1}{2} \implies \begin{cases} x=\lambda-y \\ 2=\lambda-z \end{cases} \implies x+y=2+z \implies 2+z+z=0 \implies z=-1, x+y=1 \implies$ $\begin{cases} x^2+y^2=24-z^2=23 \\ (x+y)^2=x^2+y^2+2xy=1^2=1 \end{cases} \implies xy=-11 \implies y=1-x \implies x(1-x)=-11 \implies x^2-x-11=0 \implies x=\frac{1\pm 3\sqrt{5}}{2}, y=\frac{1-3\sqrt{5}}{2}$
       * This produces the points $f\left(\frac{1+3\sqrt{5}}{2},\frac{1-3\sqrt{5}}{2},1\right)=-13$ and $f\left(\frac{1-3\sqrt{5}}{2},\frac{1+3\sqrt{5}}{2},1\right)=-13$

Reconstructing a Function from its Gradient

- If we have ∇f, how do we obtain f?

- Method 1: Integrate one of the partial derivatives, creating a "constant of integration" that's a function of the other variables; take the partial derivative with respect to the other variables, and compare against the gradient to solve for the constants of integration
- Example: $\nabla f = (1 + y^2 + xy^2)\hat{\imath} + (x^2 y + y + 2xy + 1)\hat{\jmath}$
  - $\frac{\partial f}{\partial x} = 1 + y^2 + xy^2 \implies f = x + xy^2 + \frac{1}{2}x^2 y^2 + \phi(y)$
    - * Note the constant of integration here is a "constant" with respect to x only, meaning it could be any function of y
  - Now differentiate: $\frac{\partial f}{\partial y} = 2xy + x^2 y + \phi'(y) = x^2 y + y + 2xy + 1 \implies \phi'(y) = y + 1 \implies \phi(y) = \frac{1}{2}y^2 + y + C$
  - Therefore $f(x, y) = x + xy^2 + \frac{1}{2}x^2 y + \frac{1}{2}y^2 + y + C$
- Method 2: Integrate all partial derivatives, and match the terms to get the final expression
- Example: $\nabla f = (\cos x - y \sin x)\hat{\imath} + (\cos x + z^2)\hat{\jmath} + (2yz)\hat{k}$
  - $f_x = \cos x - y \sin x \implies f = \sin x + y \cos x + \phi_1(y, z)$
  - $f_y = \cos x + z^2 \implies f = y \cos x + yz^2 + \phi_2(x, z)$
  - $f_z = 2yz \implies f = yz^2 + \phi_3(x, y)$
  - Since all 3 of these have to be true, we can conclude that $f(x, y, z) = \sin x + y \cos x + yz^2 + C$
- Not all $P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$ are gradients!
  - Example: $\nabla f(x, y) = y\hat{\imath} - x\hat{\jmath}$
    - * $f_x = y \implies f_{xy} = 1$
    - * $f_y = -x \implies f_{yx} = -1$
    - * $\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$
    - * Since mixed partials do not agree, but all derivatives are continuous, this contradicts Clairaut's theorem so f could not exist
- Theorem: Let P and Q be functions of two variables, each continuous and differentiable, then $P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$ is a gradient iff $\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y)$
  - In 3 dimensions, we need to apply this 3 times (comparing $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ and $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$), but there's no need for the third order partials

Lecture 35, Apr 8, 2022

Multivariable Optimization Example: Rocket Science

- Tsiolkovsky's Rocket Equation: $\Delta v = I_{sp} \ln \frac{m_i}{m_f}$
  - Δv is the momentum change given to the rocket
  - I_{sp} is the specific impulse of the fuel (i.e. speed of exhaust gases)
  - m_i and m_f are the initial and final mass
- Escape velocity of the Earth is about $\Delta v \approx 11$km/s; $I_{sp} \approx 3.2$km/s for common rocket fuel
- For a single stage rocket: $\frac{m_i}{m_f} = \exp\left(\frac{11}{3.2}\right) = 31.1 \implies \frac{m_{fuel}}{m_i} = \frac{m_i - m_f}{m_i} = 1 - \frac{m_f}{m_i} = 0.97$
  - 97% of the rocket would have to be fuel for the rocket to break escape velocity
  - Comparison: Aluminum beverage can is 96%
- What if we limit fuel to be 80% of the total weight?
- Break down $m_i = A + M_s + M_f$ where A is the payload, M_s is the rocket structure, M_{fuel} is the fuel
  - Define $S = \frac{M_s}{M_s + M_{fuel}}$ as the structure factor, i.e. the fraction of the mass without payload that's the structure

- Define rocket mass $M_r = M_s + M_{fuel}$
- $\dfrac{m_i}{m_f} = \dfrac{A + M_s + M_{fuel}}{A + M_s} = \dfrac{A + M_r}{A + SM_r} \implies \Delta v = I_{sp} \ln\left(\dfrac{A + M_r}{A + SM_r}\right)$
  * If we plug in $S = 0.2$ and I_{sp} from before, without any payload this would give $\Delta v = 5.15$km/s
- Given a 3-stage rocket, how can we optimize the sizes of the 3 stages to minimize the total rocket mass?
  - Let the 3 stages have masses M_1, M_2, M_3 and the same structure factor S
  - The first stage has initial mass $A + M_1 + M_2 + M_3$; after the fuel is expended, $M_1 \to SM_1$, then the second stage has mass $A + M_2 + M_3$ and so on
  - $\Delta v_{tot} = \Delta v_1 + \Delta v_2 + \Delta v_3$

$$= I_{sp}\left[\ln\left(\frac{A + M_1 + M_2 + M_3}{A + SM_1 + M_2 + M_3}\right) + \ln\left(\frac{A + M_2 + M_3}{A + SM_2 + M_3}\right) + \left(\frac{A + M_3}{A + SM_3}\right)\right]$$

- Problem: Given A, Δv, I_{sp}, and S for a 3 stage rocket, find M_1, M_2, M_3 such that $M_r = f(M_1, M_2, M_3) = M_1 + M_2 + M_3$ is minimized, subject to the constraint of the rocket equation above
  - Constraint: $g(M_1, M_2, M_3) = \ln\left(\dfrac{A + M_1 + M_2 + M_3}{A + SM_1 + M_2 + M_3}\right) + \ln\left(\dfrac{A + M_2 + M_3}{A + SM_2 + M_3}\right) + \left(\dfrac{A + M_3}{A + SM_3}\right) = \dfrac{\Delta v}{I_{sp}}$
  - $\nabla f = (1, 1, 1)$
  - $g = \ln(A + M_1 + M_2 + M_3) - \ln(A + SM_1 + M_2 + M_3) + \ln(A + M_2 + M_3) - \ln(A + SM_2 + M_3) + \ln(A + M_3) - \ln(A + SM_3)$
  - For $M_1 : \lambda\dfrac{\partial f}{\partial M_1} = \dfrac{\partial g}{\partial M_1} \implies \lambda = \dfrac{1}{A + M_1 + M_2 + M_3} - \dfrac{S}{A + SM_1 + M_2 + M_3}$
  - For $M_2 : \lambda\dfrac{\partial f}{\partial M_2} = \dfrac{\partial g}{\partial M_2} \implies \lambda = \dfrac{1}{A + M_1 + M_2 + M_3} - \dfrac{1}{A + SM_1 + M_2 + M_3} + \dfrac{1}{A + M_2 + M_3} - \dfrac{S}{A + SM_2 + M_3}$
  - For $M_3 : \lambda\dfrac{\partial f}{\partial M_3} = \dfrac{\partial g}{\partial M_3} \implies \lambda = \dfrac{1}{A + M_1 + M_2 + M_3} - \dfrac{1}{A + SM_1 + M_2 + M_3} + \dfrac{1}{A + M_2 + M_3} - \dfrac{1}{A + SM_2 + M_3} + \dfrac{1}{A + M_3} - \dfrac{S}{A + SM_3}$
  - Subtracting the third from the second equation gives: $0 = \dfrac{1}{A + SM_2 + M_3} - \dfrac{S}{A + SM_2 + M_3} + \dfrac{S}{A + SM_3} - \dfrac{1}{A + M_3} \implies -\dfrac{S}{A + SM_3} = \dfrac{1 - S}{A + SM_2 + M_3} - \dfrac{1}{A + M_3}$
    * $-\dfrac{S(A + M_3)}{A + SM_3} = \dfrac{(1 - S)(A + M_3)}{A + SM_2 + M_3} - 1 = \dfrac{A + M_3 - SA - SM_3 - A - SM_2 - M_3}{A + SM_2 + M_3} = -\dfrac{S(A + M_2 + M_3)}{A + SM_2 + M_3}$
    * $\dfrac{A + M_3}{A + SM_3} = \dfrac{A + M_2 + M_3}{A + SM_2 + M_3}$
    * Note these are the things inside ln in the second and third terms in g
  - Subtracting the second equation from the first and going through the same steps, we obtain the relation $\dfrac{A + M_2 + M_3}{A + SM_2 + M_3} + \dfrac{A + M_1 + M_2 + M_3}{A + SM_1 + M_2 + M_3}$
    * These are the arguments in ln in the first and second terms in g
  - This basically tells us that $\dfrac{m_i}{m_f}$ should be the same for each stage of the rocket
  - Let $N = \dfrac{A + M_3}{A + SM_3} \implies \dfrac{\Delta v}{I_{sp}} = 3\ln N \implies N = \exp\left(\dfrac{\Delta v}{3I_{sp}}\right)$
  - $M_3 = A\dfrac{N - 1}{1 - SN}, M_2 = (A + M_3)\dfrac{N - 1}{1 - SN}, M_1 = (A + M_2 + M_3)\dfrac{N - 1}{1 - SN}$

Lecture 36, Apr 11, 2022

Differentiability of an Integral w.r.t. Its Parameter

- Consider $F(x) = \int_c^d f(x, y)\,\mathrm{d}y$; what happens when we try to take its derivative?
- Theorem: If $f(x, y)$ has a continuous derivative with respect to x in the closed rectangle $x \in [a, b], y \in [c, d]$, then for $x \in [a, b]$, $\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \int_c^d f(x, y)\,\mathrm{d}y = \int_c^d \frac{\partial f}{\partial x}\,\mathrm{d}y$
  - Proof: Given $x, x + h \in [a, b]$, then:
    * $\begin{aligned}\frac{F(x+h) - F(x)}{h} &= \frac{1}{h}\int_c^d f(x+h, y)\,\mathrm{d}y - \frac{1}{h}\int_c^d f(x, y)\,\mathrm{d}y \\ &= \frac{1}{h}\int_c^d f(x+h, y) - f(x, y)\,\mathrm{d}y\end{aligned}$
    * $\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0}\int_c^d \frac{f(x+h, y) - f(x, y)}{h}\,\mathrm{d}y \\ &= \int_c^d \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}\,\mathrm{d}y \\ &= \int_c^d \frac{\partial f}{\partial x}\,\mathrm{d}y\end{aligned}$
    * We can bring the limit inside the integral, because the integral is a sum and limit laws allow us to distribute the limit over sums
- Example: $F(x) = \int_2^4 e^{xy}\,\mathrm{d}y$
  - Doing the integral first: $F(x) = \left[\frac{e^{xy}}{x}\right]_2^4 = \frac{e^{4x} - e^{2x}}{x}$
    * $\frac{\mathrm{d}F}{\mathrm{d}x} = e^{4x}\left(\frac{4x - 1}{x^2}\right) - e^{2x}\left(\frac{2x - 1}{x^2}\right)$
  - Using the theorem: $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_2^4 \frac{\partial}{\partial x} e^{xy}\,\mathrm{d}x = \int_2^4 y e^{xy}\,\mathrm{d}y$
    * Using integration by parts: $\left[\frac{y}{x}e^{xy}\right]_2^4 - \int_2^4 \frac{e^{xy}}{x}\,\mathrm{d}y = \left[\left(\frac{y}{x} - \frac{1}{x^2}\right)e^{xy}\right]_2^4 = e^{4x}\left(\frac{4x - 1}{x^2}\right) - e^{2x}\left(\frac{2x - 1}{x^2}\right)$
- Consider $A(t) = \int_{x_1(t)}^{x_2(t)} f(x)\,\mathrm{d}x$ and note $\frac{\mathrm{d}A}{\mathrm{d}t} = f(x_2(t))\frac{\mathrm{d}x_2}{\mathrm{d}t} - f(x_1(t))\frac{\mathrm{d}x_1}{\mathrm{d}t}$ by the chain rule and FTC
- Theorem: Leibniz's Rule: Given a region R in the xy plane in which $\phi_1(x), \phi_2(x)$ and $f(x, y)$ are continuously differentiable, if $F(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,\mathrm{d}y$, then $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}\,\mathrm{d}y + f(x, \phi_2(x))\frac{\mathrm{d}\phi_2}{\mathrm{d}x} - f(x, \phi_1(x))\frac{\mathrm{d}\phi_1}{\mathrm{d}x}$
- Example: $F(x) = \int_0^{x^2} \sin(xy)\,\mathrm{d}y$
  - $f(x, y) = \sin(xy), \phi_1(x) = 0, \phi_2(x) = x^2$
  - $\frac{\partial f}{\partial x} = y\cos(xy), \frac{\mathrm{d}\phi_1}{\mathrm{d}x} = 0, \frac{\mathrm{d}\phi_2}{\mathrm{d}x} = 2x$
  - $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_0^{x^2} y\cos(xy)\,\mathrm{d}y + 2x\sin(x \cdot x^2) + 0\sin(0) = \int_0^{x^2} y\cos(xy)\,\mathrm{d}y + 2x\sin(x^3)$
- This can be used as an integration technique (Feynman integration)

- Example: $F(x) = \int_0^1 \frac{y^x - 1}{\ln y} \, \mathrm{d}y$ for $x > -1$
  - $\frac{\mathrm{d}F}{\mathrm{d}x} = \int_0^1 \frac{\partial}{\partial x} \left(\frac{y^x - 1}{\ln y} \right) \, \mathrm{d}y$
    $= \int_0^1 \frac{y^x \ln y}{\ln y} \, \mathrm{d}y$
    $= \int_0^1 y^x \, \mathrm{d}y$
    $= \left[\frac{y^{x+1}}{x+1} \right]_0^1$
    $= \frac{1}{x+1}$
  - $F(x) = \int \frac{1}{x+1} \, \mathrm{d}x = \ln|x+1| + C$
  - From the original expression $F(0) = \int_0^1 \frac{y^0 - 1}{\ln y} \, \mathrm{d}y = 0 \implies C = 0 \implies F(x) = \ln(x+1)$