Lectures 1/2, Jan 14/17, 2022

Electric Variables

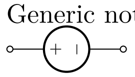
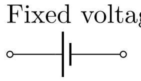
- An electric circuit is an interconnection of conductors, nonconductors and semiconductors
- The flow of electricity always involves the movement of charge
- Fundamental electric variables:
  1. Electric current
     - If we take a cross section of a conductor with moving charges, we get charge $q(t)$ as a function of time
     - Define *current* as the rate of change charge with respect to time, $i \equiv \frac{\mathrm{d}q}{\mathrm{d}t}$ with units of C/s = A (Coulombs per second, or Amperes)
     - Current also has a direction (i.e. the direction of charge flow); the convention is the direction of movement of the *positive* charge (even though negative charges is what's actually moving physically)
     - Direction shown with arrows
  2. Voltage
     - Movement of charge is associated with energy
     - Define *voltage* between two points as the energy required to move 1 Coulomb of charge between two points in a circuit
     - $v \equiv \frac{\mathrm{d}w}{\mathrm{d}q}$ where w is energy, q, is charge; units of J/C = V (Joules per Coulomb, or Volts)
     - Voltage also has a *polarity* (positive or negative); the positive side is where the movement starts, and the negative side is where the movement ends
       * When we say "the voltage between point A and point B", point A is the positive side and point B is the negative side
     - Polarity shown with positive and negative signs
  3. Power
     - Rate of absorbing or delivering energy with respect to time
     - $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}t} \implies P \equiv \frac{\mathrm{d}w}{\mathrm{d}t} = vi$ with units of J/s = W (Joules per second or Watts)
     - To differentiate whether power is consumed or generated, we need another sign convention
     - Passive sign convention (PSC): for a pair of v and i, PSC holds if current enters the positive side of the voltage polarity first
       * If PSC holds, then $P = +vi$; $P > 0 \implies P$ is absorbed; $P < 0 \implies P$ is delivered
       * Otherwise, $P = -vi$; same holds for the meaning of sign of P

Lecture 3, Jan 19, 2022

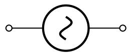
Power Conservation

- For any circuit, $\sum P_k = 0$ (power conservation law)
  - Note the signs are very important here

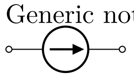
Circuit Elements – Independent Sources

1. Independent voltage sources: voltage sources that provide a fixed voltage no matter what current is flowing through it
   - The voltage could be fixed or a time-variant function, e.g. $v_s(t) = 5\cos(100t + 2)\mathrm{V}$
   - Generic notation:
     
   - Fixed voltages:
     

    - Sinusoidal voltage source:

    - These are just *models*; they don't actually exist because in reality current always impact the voltage a little bit
    - Which way the current flows depends on the circuit, so whether the voltage source produces or consumes power depends on the circuits
2. Independent current sources: current sources that provide a fixed current no matter the voltage across it
    - Current could be fixed or time-variant
    - Generic notation:

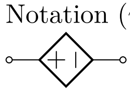
      

    - Polarity depends on the rest of the circuit and so does whether it generates or consumes power

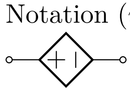
Lecture 4, Jan 24, 2022

Circuit Elements – Dependent Sources

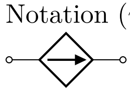
- Linear dependent sources:
    1. Voltage-dependent voltage source (controlled voltage source): voltage provided by the source is kv_x where v_x is the voltage somewhere in the circuit to which this source is connected
        - The voltage doesn't dependent on the current that passes through; it depends on the voltage somewhere else in the circuit completely
        - Notation (v_x marked in the circuit):

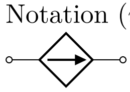
          kv_x
        - k is dimensionless (voltage to voltage)
    2. Current-dependent voltage source: voltage provided is ki_x, like the voltage-dependent voltage source but for current
        - Notation (i_x marked in the circuit):

          ki_x
        - k has dimensions of voltage over current, V/A
    3. Voltage-dependent current source: current output is kv_x
        - Notation (v_x marked in the circuit):

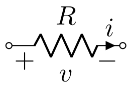
          kv_x
        - k has dimensions of current over voltage, A/V
    4. Current-dependent current source: current output is ki_x
        - Notation (i_x marked in the circuit):

          ki_x
        - k is dimensionless (current to current)
- Just like independent sources, perfectly linear dependent sources don't exist in the real world, but under certain conditions we can use them to model real things

Other Circuit Elements

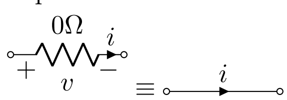
- Resistors: ratio of voltage over current is always a constant, $\frac{v}{i} = R$
  - Notation:

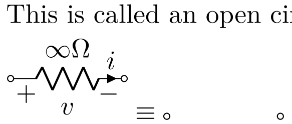


  - The relation $\frac{v}{i} = R$ is only true when PSC holds (when it doesn't, we need a minus sign)
  - R has units of A/V = Ω (Ohm)
  - Alternatively, $G = \frac{1}{R}$ is the *conductance* (as opposed to R being the *resistance*), which has units of $\Omega^{-1} = \mho$ (mho) or Siemens Si
  - Assuming R is positive, power going through will always be positive, i.e. the resistor always consumes power

Lecture 5, Jan 26, 2022

Resistors Continued

- Extreme cases of Ohm's law
  1. $R = 0 \implies \forall i, v = 0$; this is called a short circuit
     - A path with zero resistance is called an ideal conductor
     - 

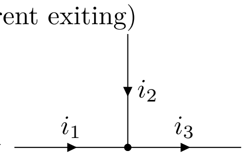
     - All parts of a circuit connected by ideal conductors can be considered the same node in a circuit
  2. $R \to \infty \implies \forall v, i = 0$
     - This is called an open circuit
     - 

Structure of a Circuit

- Node: A junction of two or more circuit elements
- Path: Start from one node, and if no other node is passed through more than once except the first one which may be passed twice, this is a path
- Loop: A path that begins and ends at the same node that consists of at least 3 nodes

Lecture 6, Jan 28, 2022

Circuit Analysis Laws

- Kirchhoff's Current Law (KCL): The algebraic sum of the currents entering a node is zero (or the current exiting)
  - 

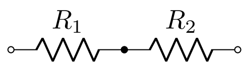
  - If we assume current entering node is positive, then i_1 and i_2 are positive, i_3 is negative, therefore $i_1 + i_2 - i_3 = 0 \implies i_3 = i_1 + i_2$
  - We can also assume current leaving the node is positive, so $-i_1 - i_2 + i_3 = 0$ which gets us the same relation

  - Alternatively can be stated as "sum of current entering the node equals sum of current leaving the node"
- Kirchhoff's Voltage Law (KVL): The algebraic sum of the voltages in a loop are zero
  - The dual of the KCL
  - Note the sign changes depending on which direction you're going

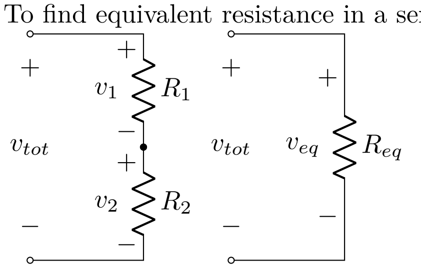
Lecture 9, Feb 2, 2022

Simplifying Series and Parallel Resistors

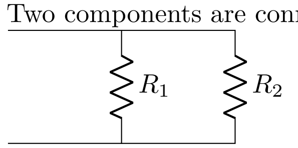
- Two components are connected *in series* if they're connected back-to-back, and at the point of connection there is no other current path:



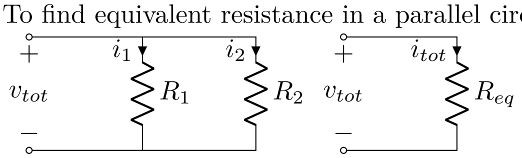
- To find equivalent resistance in a series circuit, compare:



  - KVL gives: $-v_{tot} + v_1 + v_2 = 0$
    * Applying Ohm's law: $v_1 = R_1 i_{tot}$ and $v_2 = R_2 i_{tot}$
    * Substituting the voltages back in: $-v_{tot} + R_1 i_{tot} + R_2 i_{tot} = 0 \implies v_{tot} = (R_1 + R_2) i_{tot}$
    * Compare this to Ohm's law for the second circuit, we see that the equivalent resistance is $R_1 + R_2$
    * This generalizes to any number of resistors to give $R_{eq} = R_1 + R_2 + \cdots + R_n$
    * In the extreme cases, if one resistor is an open circuit $R = \infty$, the entire circuit can be considered as an open connection; if one resistor is a short circuit, then it wouldn't have any effect
- Two components are connected *in parallel* if they share two common nodes:



- To find equivalent resistance in a parallel circuit, compare:



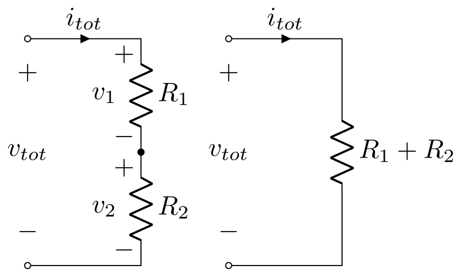
  - KCL gives: $i_{tot} = i_1 + i_2$
    * KVL gives: $v_{tot} = R_i i_1 = R_2 i_2 \implies i_1 = \dfrac{v_{tot}}{R_1}, i_2 = \dfrac{v_{tot}}{R_2}$
    * $i_{tot} = \dfrac{v_{tot}}{R_1} + \dfrac{v_{tot}}{R_2}$
    * Compare this to circuit 2 we get $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} \implies R_{eq} = \dfrac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \dfrac{R_1 R_2}{R_1 + R_2}$
      - Alternatively, the equivalent conductance of two resistors in parallel is the sum of the conductances
    * The conductance relation generalizes to any number of resistors; however $\dfrac{R_1 R_2}{R_1 + R_2}$ becomes $\dfrac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$ for 3 resistors, $\dfrac{R_1 R_2 R_3 R_4}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4}$ and so on

* In the extreme cases, if one resistor is a short circuit, then the entire circuit can be considered a short circuit; if one resistor is an open connection, then it does not have an impact (since $\frac{1}{R} \to 0$)

Lecture 10, Feb 2, 2022

Voltage Division

- Consider 2 resistors in series:

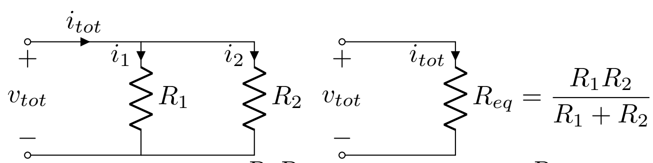


  - We want to know how the voltage v_{tot} is divided between the two resistors
    * KVL gives: $v_{tot} = v_1 + v_2$
    * Ohm's law gives $i_{tot} = \frac{v_1}{R_1} = \frac{v_{tot}}{R_1 + R_2} \implies v_1 = \frac{R_1}{R_1 + R_2} v_{tot}$
- Voltage drop across a resistor in a series circuit is $\frac{R}{R_{tot}}$ times the total voltage drop (note the polarities have to match)
  - If the polarity of the resistor matches the polarity of v_{tot} then the relation works; if it's opposite then we get the voltage negative instead

Lecture 11, Feb 4, 2022

Current Division

- Similar rule can be found for current in a parallel circuit:

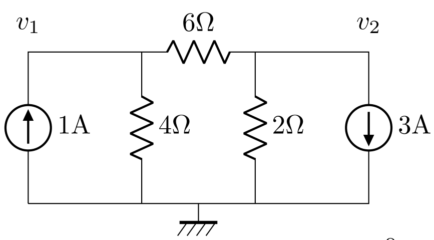


  - $v_{tot} = R_1 i_1 = \frac{R_1 R_2}{R_1 + R_2} i_{tot} \implies i_1 = \frac{R_2}{R_1 + R_2} i_{tot}$
- The current division principle is the dual of the voltage division principle; note the current division ratio uses the resistances of the **other** branches ($\frac{R_2}{R_1 + R_2}$ for current, $\frac{R_1}{R_1 + R_2}$ for voltage)
- As with voltage division, the signs only work if the direction of i_1 matches the direction of i_{tot}; if the directions don't match, we need an additional negative sign
- We can write this in terms of the conductance as $i_1 = \frac{G_1}{G_1 + G_2} i_{tot}$, similar to the voltage law
- For multiple resistors in series, we can either use the conductances, or collapse the other resistors down to a single resistor; for 3 resistors it becomes $i_1 = \frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} i_{tot}$ and so on
- Can also be written as $i_1 = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} i_{tot}$

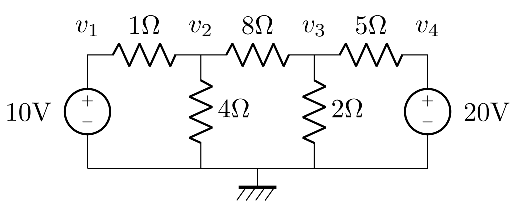
Lecture 12, Feb 7, 2022

Nodal Analysis

- Nodal analysis is an algorithmic method for circuit analysis; it finds the node voltages at every node in the circuit
  - Define *node voltage* as the voltage between a node and a reference point (common ground), with positive polarity at the node and negative polarity at the reference point
  - The reference (ground) node is typically denoted with a ground symbol: ⏚ or ⏚
- Apply KCL for every node in terms of the node voltages
  - Voltage between two nodes is the difference of their nodal voltages
  - $v_{AB} = v_A - v_B$ and $v_{BA} = v_B - v_A$
- Procedure:
  1. Find all the nodes in the circuit and label them, choose one as ground
     - Choice of ground node is arbitrary but sometimes it can simplify the math
     - Choose the node that's connected to the highest number of voltage sources; prefer independent sources over dependent sources
  2. Assume current directions/signs (negative for current entering node, positive for current leaving)
  3. Write KCL for all the ungrounded notes
     - Current sources: we have voltage directly, resistors: use Ohm's law
     - If we have a voltage source between the ground node and another node, we can get the voltage of that node directly
     - Usually we always write the current that leaves a node via a resistor since it gives a positive sign
  4. Solve the system for the nodal voltages and use the nodal voltages to find anything else needed
- Example circuit 1:



  - KCL at node 1: $-1 + \frac{v_1 - 0}{4} + \frac{v_1 - v_2}{6} = 0$
    * KCL at node 2: $3 + \frac{v_2 - 0}{2} + \frac{v_2 - v_1}{6} = 0$
    * Solution: $v_1 = \frac{2}{3}\text{V}, v_2 = -\frac{13}{2}\text{V}$
- Example circuit 2:

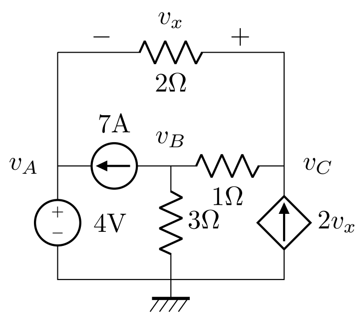


  - We get $v_1 = 10\text{V}, v_2 = 20\text{V}$ immediately
    * Node 2: $\frac{v_2 - v_1}{1} + \frac{v_2 - 0}{4} + \frac{v_2 - v_3}{8} = 0$
    * Node 3: $\frac{v_3 - v_2}{8} + \frac{v_3 - 0}{2} + \frac{v_3 - v_4}{5} = 0$
    * Solution: $v_2 = 7.82\text{V}, v_3 = 6.03\text{V}$

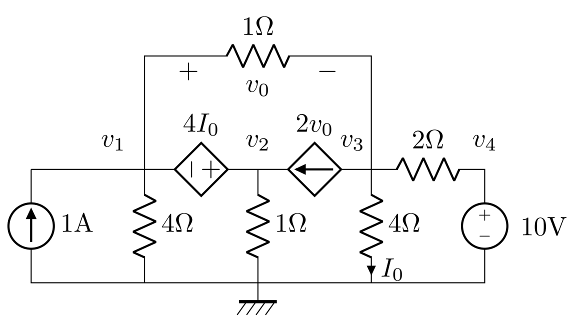
Lecture 13, Feb 9, 2022

Nodal Analysis With Voltage Sources

- When there is a dependent source, express the parameter that the source depends on in terms of nodal voltages
- When there is an ungrounded voltage source, instead of writing KCL for the two nodes separately, write KCL for the "supernode" that combines the two nodes connected by the source
  - This reduces the number of equations by 1, but we can get this equation back by relating the voltages of the two nodes using the voltage source; inside the supernode one node is kept at a higher voltage than the other by the source
  - This extra equation is called the *supplementary equation* for the supernode
- Example circuit 1:



  - $v_A = 4\text{V}$
  - In this case, $v_x = v_C - v_A$
  - At node B: $7 + \frac{v_B}{3} + \frac{v_B - v_C}{1} = 0$
  - At node C: $\frac{v_C - v_A}{2} + \frac{v_C - v_B}{1} - 2(v_C - v_A) = 0$
- Example circuit 2:

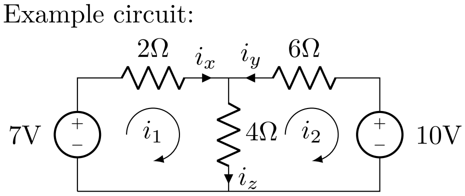


  - $v_4 = 10\text{V}$
  - $I_0 = \frac{v_3}{4}, v_0 = v_1 - v_3$
  - Combine nodes 1 and 2 into a supernode: $-1 + \frac{v_1}{4} + \frac{v_1 - v_3}{1} + \frac{v_2}{1} - 2(v_1 - v_3) = 0$
  - At node 3: $2(v_1 - v_3) + \frac{v_3 - v_1}{1} + \frac{v_3}{4} + \frac{v_3 - v_4}{2} = 0$
  - 2 equations, 3 unknowns (v_1, v_2, v_3); the last equation comes from the voltage source between nodes 1 and 2, producing $v_2 - v_1 = 4\frac{v_3}{4}$

Lecture 14, Feb 11, 2022

Mesh Analysis

- When there are a lot of components connected in series, there are a lot of nodes so nodal analysis is not as efficient computationally
- Nodal analysis is preferred when there are lots of parallel connections since there are fewer nodes; mesh analysis is preferred when there are lots of series connections since there are fewer meshes
- Definition: A mesh is a type of loop that does not have any other loop inside it
- In mesh analysis the objective is finding all the mesh currents; with the mesh currents we can find all voltages and currents
  - We do this by writing KVL for all the meshes in terms of mesh currents
- Steps for mesh analysis:
  1. Identify all meshes in the circuit
  2. Associate a circulating current with each mesh (the mesh current)
     - These currents are hypothetical currents that can have any direction (commonly clockwise, stick to one direction to reduce mistakes)
     - These are not branch currents (which are the real currents through the branches)
     - Branch currents can be expressed in terms of mesh currents by adding all the mesh currents that pass through the branch with the right sign
  3. Write KVL for all the meshes in terms of the mesh voltages
  4. Solve for all mesh currents and use the mesh currents to solve for voltages and currents as needed
- Example circuit:



  - Two meshes (the 2 inner loops)
  - $i_x = i_1, i_y = -i_2, i_z = i_1 - i_2$
  - For mesh 1: $-7 + 2i_i + 4(i_1 - i_2) = 0$
  - For mesh 2: $4(i_2 - i_1) + 6i_2 + 10 = 0$
  - Now we can solve for i_1 and i_2

Lecture 15, Feb 14, 2022

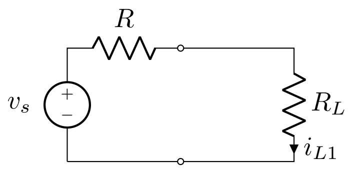
Mesh Analysis With Current Sources

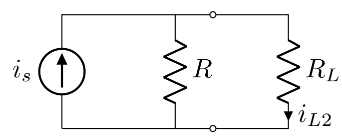
- When there's a current source at the edge of a circuit, we can get the branch current from the source directly
- For dependent current (or voltage sources), express the variable that it depends on in terms of mesh currents, similar to nodal analysis
- When there's a current source shared by two meshes, combine the two meshes into a single "supermesh" like a supernode for nodal analysis and write KVL for this supermesh
  - This takes away one equation, but another equation can be written for the current source
- Mesh analysis can only solve *planar* circuits (all the circuits we see in this course are planar)

Lecture 16, Feb 16, 2022

Source Transformation

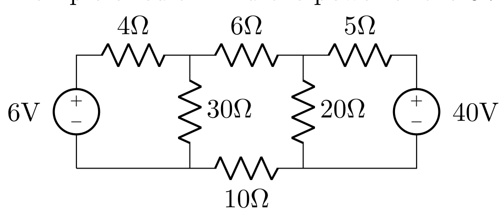
- This method is not as strong as nodal analysis or mesh analysis, but gives insight into circuit analysis
- Consider 2 circuits:



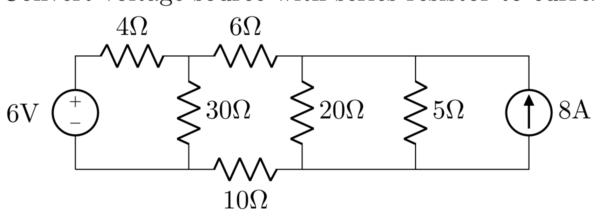


- For circuit 1, $i_{L1} = \dfrac{v_s}{R + R_L}$ by Ohm's law
    * For circuit 2, $i_{L2} = i_s \dfrac{R}{R + R_L}$ by current division
    * If both circuits are equivalent, then we'll get $i_{L1} = i_{L2} \implies \dfrac{v_s}{R + R_L} = i_s \dfrac{R}{R + R_L} \implies v_s = i_s R$

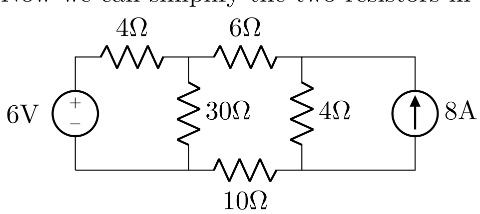
- Source transformation: We can transform a voltage source with a resistor in series to a current source with a resistor in parallel if $v_s = R i_s$
    - The direction of the sources do not follow PSC
- Example circuit: Find the power of the 6V source:



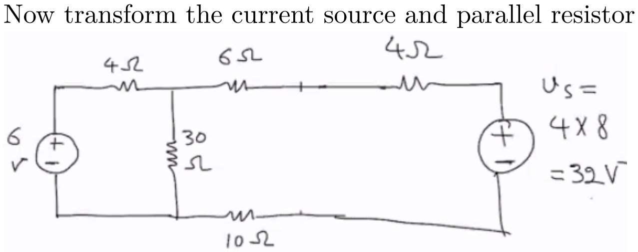
- Convert voltage source with series resistor to current source with parallel resistor:



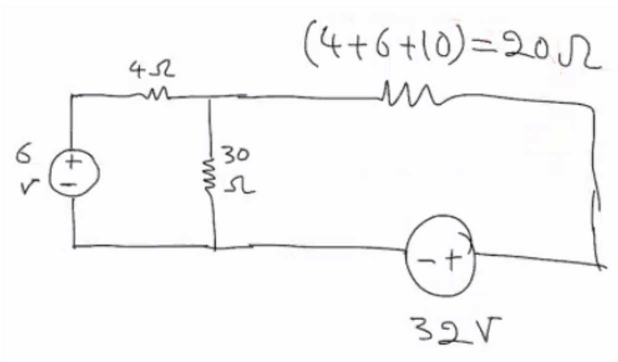
- Now we can simplify the two resistors in parallel on the right side:



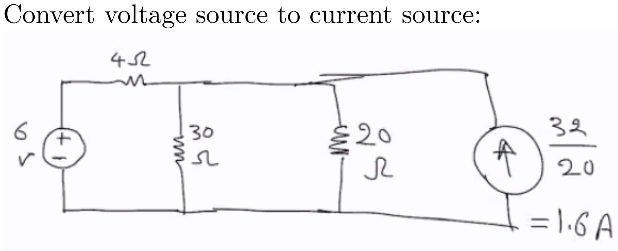
- Now transform the current source and parallel resistor to a voltage source and a series resistor:



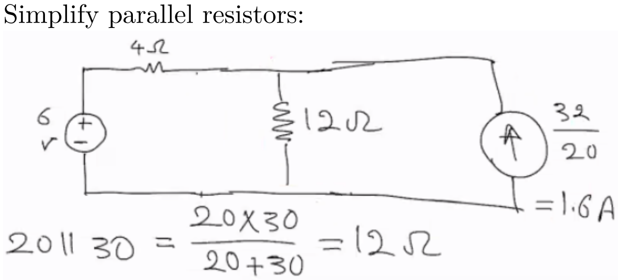
- Simplify series resistors:



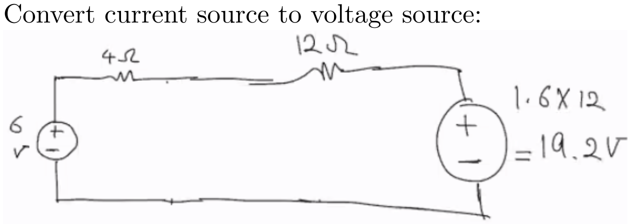
– Convert voltage source to current source:



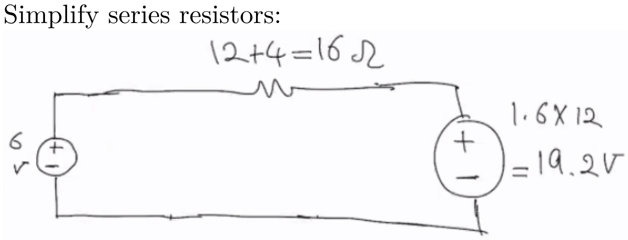
– Simplify parallel resistors:



– Convert current source to voltage source:



– Simplify series resistors:

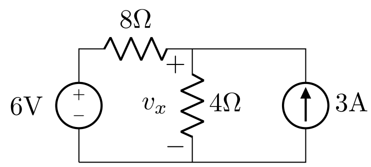


  – Now we can use KVL on this loop to find the current in this circuit and get the power
- If we have a voltage source in parallel with a resistor, the value of the resistance does not affect the voltage distributed to the rest of the circuit, so it doesn't impact the rest of the circuit
  – Since the resistor has no affect on the rest of the circuit, we can remove it altogether (open the circuit)
- If we have a current source in series with a resistor, the value of the resistance does not affect the current distributed to the rest of the circuit, so it has no impact either
  – Since it has no affect, we can remove it (short the circuit)

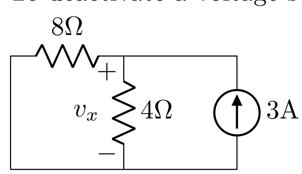
Lecture 17, Feb 18, 2022

Superposition Principle

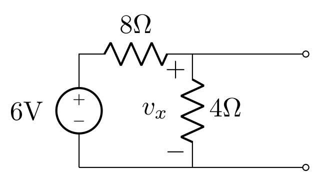
- Linear circuit: A circuit that consists of independent sources, linear dependent sources, and linear elements
  - Examples of linear elements include resistors, capacitors and inductors
- Superposition principle: The response of a linear circuit to multiple independent sources is equal to the algebraic sum of the responses caused by each independent source acting alone
  - This allows us to look at only one independent source at a time to simplify the problem
- Example circuit: Find the voltage v_x:



  - Phase 1: Deactivate the voltage source
    * To deactivate a voltage source, we short it out so the voltage is zero:



    * Now the resistors are in parallel; use current division: $i_{x_1} = 3\dfrac{8}{4+8} = 2\text{A}$, so $v_{x_1} = 4i_{x_1} = 8\text{V}$
  - Phase 2: Deactivate the current source
    * To deactivate a current source, we open the circuit so the current is zero:



    * Now the resistors are in series; use voltage division: $v_{x_2} = 6\dfrac{4}{4+8} = 2\text{V}$
  - The voltage across v_x with the two sources combined is $v_x = v_{x_1} + v_{x_2} = 10\text{V}$

Lecture 18, Feb 28, 2022

Thevenin Equivalent Circuit

- Equivalent circuits allow us to simplify parts of circuits so we still get the same behaviour elsewhere
- Thevenin's Theorem: A linear circuit can be replaced by a series connection of a voltage source (Thevenin voltage) and a resistor (Thevenin resistance) (Thevenin equivalent circuit), to give the same current and voltage outside the circuit
  - This generalizes equivalent resistances and source transformation to an equivalent circuit of any linear element
- The Thevenin voltage is the same as the open circuit voltage between the terminals
  - i.e. remove the load (rest of the circuit) and open the circuit, find the voltage this way and that is the Thevenin voltage
  - The open circuit voltage can be found using any circuit analysis technique (e.g. nodal/mesh analysis)
  - The polarity of the voltage source must match that of the open circuit voltage found
- The Thevenin resistance can be found through 3 different methods:
  1. If the circuit does not include a dependent source (i.e. only resistors and independent sources):

deactivate all the *independent* sources (short voltage sources, open current sources); the equivalent resistance is the Thevenin resistance

2. If the circuit includes at least 1 independent source: find the open circuit voltage V_{oc} and short circuit current i_{sc}; then $R_{Th} = \dfrac{V_{oc}}{i_{sc}}$ (V_{oc} and i_{sc} must have directions conforming to PSC)
   - This method is essentially based on source transformation; we find the Thevenin and Norton voltage/current and use source transformation to relate the two by the Thevenin/Norton resistance
3. Otherwise (applies to any linear circuit):
   1. Deactivate all *independent* sources
   2. Add a test current source of i_T between terminals
   3. Find the voltage across the current source v_T, *not* conforming to PSC
   4. $R_{Th} = \dfrac{v_T}{i_T}$

- Alternatively, don't deactivate any sources, connect a current source of I_T and find voltage V_T across it; then $V_T = MI_T + N$, and $N = V_{Th}, M = R_{Th}$
  - This allows you to find both the Thevenin voltage and resistance by solving just one circuit, but you have to work with I_T as an unknown

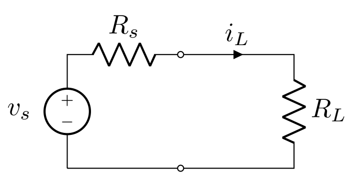
Lecture 19, Mar 2, 2022

Norton Equivalent Circuit

- The Norton Equivalent Circuit is the dual of the Thevenin Equivalent Circuit; instead of a voltage source in series with a resistor, in a Norton Equivalent Circuit the elements are replaced by a current source in parallel with a resistor
  - Thevenin and Norton circuits can be transformed into each other via source transformation
  - The Norton resistance is the same as the Thevenin resistance: $R_N = R_{Th}$
  - The Norton current can be obtained by $I_N = \dfrac{V_{Th}}{R_{Th}}$ via source transformation
  - Alternatively, short the terminals, and then I_N is the current flowing through this short
- If using a short circuit to find the Norton current, the direction of the current source must *complete the loop* with the short circuit current

Lecture 20, Mar 4, 2022

Maximum Power Transfer

- 

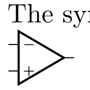
- Consider a voltage source connected to a resistor in series and then connected to a load; how do we extract maximum power from this voltage source? What R_L maximizes power?
  - $P_L = R_L i_L^2$
  - $i_L = \dfrac{v_s}{R_s + R_L} \implies P_L = \dfrac{R_L v_s^2}{(R_s + R_L)^2}$
  - To maximize P_L we differentiate it
  - $\dfrac{\mathrm{d}P_L}{\mathrm{d}R_L} = \dfrac{v_s^2(R_L + R_s)^2 - 2(R_L + R_s)R_L v_s^2}{(R_L + R_s)^4} = \dfrac{v_s^2((R_L + R_s) - 2R_L)}{(R_L + R_s)^3} = 0$
  - The only way for the derivative to equal zero is if $R_L + R_s = 2R_L \implies R_L = R_s$
  - Plugging in $R_L = R_s \implies P_{L_{max}} = \dfrac{v_s^2}{4R_s^2} = \dfrac{v_s^2}{4R_L}$

- For a voltage source and resistor, max power transfer is achieved when the load resistance equals the resistance attached to the voltage source, with max power being $\frac{v_s^2}{4R_s}$
  - For any complicated circuit we can find its Thevenin equivalent and turn it into a voltage source with resistor in series

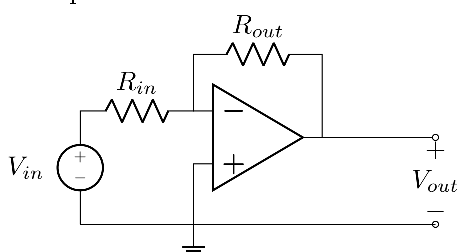
Lecture 21, Mar 7, 2022

Operational Amplifiers (Op-Amps)

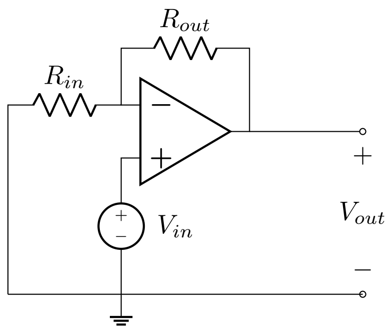
- Op amps have 3 terminals that are relevant for this course: the positive (non-inverting) terminal, the negative (inverting) terminal, and then output terminal
  - Inverting input has voltage v_1 and current i_2, non-inverting input has voltage v_2 and current i_2; output has voltage v_{out} and current i_{out} (voltages measured wrt ground)
  - The symbol is



- If there is a path between the inverting input and output (either a short or any resistance), then the op amp as a *negative feedback connection*
  - Under a negative feedback connection, $v_1 = v_2$, i.e. it forces 2 voltages to be the same (under ideal conditions), and $i_1 = i_2 = 0$ (infinite input impedance)
- In this course we only discuss ideal op amps
- Example circuit:



  - The path from the output to the inverting input forces $v_2 = v_1$, therefore the current through R_{in} is $\frac{V_{in}}{R_{in}}$
    * No current goes into the op amp so current through R_f is also I_{in}
    * Since $v_2 = 0$ the current through R_f is also $-\frac{V_{out}}{R_f}$
    * Equating these currents: $\frac{V_{in}}{R_{in}} = I_{in} = \frac{V_{out}}{R_f} \implies \frac{V_{out}}{V_{in}} = -\frac{R_f}{R_{in}}$
    * This circuit is called an *inverting amplifier* since it switches the polarity, and amplifies it by a gain of $-\frac{R_f}{R_{in}}$
- If the input voltage now goes into the noninverting input, the output voltage is no longer inverted, so this is now a *noninverting amplifier*:

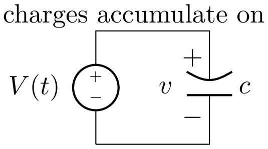


- In general, to solve an op amp circuit when there is a connection from output to inverting input, first try to find either v_1 or v_2, and then use the relationship $v_1 = v_2$ to find the voltage at the other terminal, and then solve the rest of the circuit

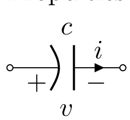
Lecture 22, Mar 9, 2022

Capacitors

- A capacitor consists of 2 conducting plates separated by an insulator; when connected to a voltage, charges accumulate on the plates, creating an electric field and storing energy:

  - The accumulated charge is proportional to the voltage: $q(t) = cv(t)$
    * c is the *capacitance*, and is determined by the physical characteristics of the capacitor (similar to resistance for a resistor)
    * Capacitance has units of Coulombs per volt or farads: $\mathrm{C/V} = \mathrm{F}$
      - In practice one farad is a very large capacitance; most capacitors are in the order of microfarads or smaller
- Translating this into current: $i = \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}cv(t) \implies i(t) = c\dfrac{\mathrm{d}v}{\mathrm{d}t}$
  - Current passing through a capacitor is proportional to the rate of change of voltage
  - This relation holds if PSC holds; otherwise $i = -c\dfrac{\mathrm{d}v}{\mathrm{d}t}$
- In the other direction: $\displaystyle\int_{t_1}^{t_2} c\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_1}^{t_2} i(t)\,\mathrm{d}t \implies v(t_2) - v(t_1) = \frac{1}{c}\int_{t_1}^{t_2} i(t)\,\mathrm{d}t$
  - $\displaystyle v(t) = v(0) + \frac{1}{c}\int_0^t i(\tau)\,\mathrm{d}\tau$
  - To find the voltage of a capacitor at time t, integrate the current
  - We need both the current function and a known value of $v(t)$, unlike with current from voltage where we only need the voltage function
- Properties of capacitors:

  1. If the voltage is constant (i.e. DC), then current is always 0, since $\dfrac{\mathrm{d}v}{\mathrm{d}t}$ is 0
     - A capacitor can be modelled as an open circuit in a DC circuit
  2. The voltage of a capacitor cannot change abruptly; a discontinuity in voltage creates an infinite $\dfrac{\mathrm{d}v}{\mathrm{d}t}$ and infinite current

- Find energy of a capacitor: $$\begin{aligned} W(t_2) - W(t_1) &= \int_{t_1}^{t_2} P(t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} v(t)i(t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} cv(t)\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t \\ &= c\int_{t_1}^{t_2} v\,\mathrm{d}v \\ &= \frac{1}{2}c(v^2(t_2) - v^2(t_1)) \end{aligned}$$
  - Assuming capacitor is unchanged at $t = 0$ (i.e. $v(0) = 0$), $W(t) = \frac{1}{2}cv^2(t)$
  - An ideal capacitor does not dissipate energy; it only stores and delivers energy
- Although an ideal capacitor stops all DC current, a physical capacitor has some leakage current
- A real capacitor can be modelled as an ideal capacitor in parallel with a *leakage resistance* of R_L, typically in the hundreds of megaohms

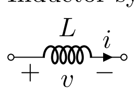
Lecture 23, Mar 11, 2022

Series and Parallel Connections of Capacitors

- Suppose we have n capacitors $c_1, \cdots, c_n$ connected in parallel
  - Each capacitor has a current $i_k = c_k\frac{\mathrm{d}v_k}{\mathrm{d}t}$ but the capacitors all have the same voltage, so $i_k = c_k\frac{\mathrm{d}v}{\mathrm{d}t}$
  - To find the equivalent capacitance, we need to find the total current
  - $i_{tot} = \sum_{k=1}^{n} i_k = \frac{\mathrm{d}v}{\mathrm{d}t}\sum_{k=1}^{n} c_k \implies c_{eq} = \sum_{k=1}^{n} c_k$
- The equivalent capacitance for capacitors in parallel is the sum of all the capacitances
- Suppose we have n capacitors $c_1, \cdots, c_n$ connected in series
  - All capacitors have the same current and each has a voltage v_k
  - KVL gives $v_{tot} = \sum_{k=1}^{n} v_k \implies \frac{\mathrm{d}v_{tot}}{\mathrm{d}t} = \sum_{k=1}^{n}\frac{\mathrm{d}v_k}{\mathrm{d}t} = \sum_{k=1}^{n}\frac{1}{c_k}i = i\sum_{k=1}^{n}\frac{1}{c_k} \implies \frac{1}{c_{eq}} = \sum_{k=1}^{n}\frac{1}{c_k}$
- The equivalent capacitance for capacitors in series is the reciprocal of the sum of the reciprocals of the capacitances
  - For c_1 and c_2 in series, $c_{eq} = \frac{c_1c_2}{c_1 + c_2}$
- The behaviour in series vs parallel for capacitors is opposite that of resistors

Inductors

- An inductor consists of a coil of conducting wire with a core of any material
- Like a capacitor, an inductor stores energy, this time in a magnetic field generated as current passes through it
  - Since the energy density for a magnetic field is much larger than that of an electric field, the energy that can be stored in an inductor is much larger than a capacitor
- Inductor symbol:



- For an inductor, voltage is related to current by $v = L\frac{\mathrm{d}i}{\mathrm{d}t}$; an inductor is the dual of a capacitor
  - L is the *inductance* and has units of henry H
    * The larger the value of L, the more energy can be stored in the inductor
    * L depends on the kind of core used in the inductor

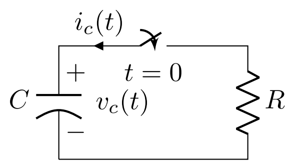
- This relation only holds if PSC holds

- To get current from voltage we can integrate: $i(t_2) = i(t_1) + \dfrac{1}{L}\displaystyle\int_{t_1}^{t_2} v \,\mathrm{d}t$ or $i(t) = i(0) + \dfrac{1}{L}\displaystyle\int_0^t v(\tau)\,\mathrm{d}\tau$
- Properties of inductors:
  1. If the current is constant, then the voltage is always 0
     - In a DC circuit (in steady state) the inductor can be modelled by a short
  2. The current of an inductor cannot change abruptly since that would create an infinite voltage
- Energy of a capacitor: $$\begin{aligned} W(t_2) - W(t_1) &= \int_{t_1}^{t_2} P(t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} v(t)i(t)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} Li(t)\frac{\mathrm{d}i}{\mathrm{d}t}\,\mathrm{d}t \\ &= L\int_{t_1}^{t_2} i\,\mathrm{d}i \\ &= \frac{1}{2}L(i^2(t_2) - i^2(t_1)) \end{aligned}$$
  - Assuming no magnetic field at $t = 0$, $W(t) = \dfrac{1}{2}Li^2(t)$
  - Like an ideal capacitor, an ideal inductor does not dissipate energy and only stores it
- The equivalent inductance of inductors in series is the sum of the inductances; in parallel it's the reciprocal of the sum of the reciprocals (like resistors)

Lecture 24, Mar 14, 2022

Source-Free RC Circuits

- In a first-order transient circuit, the relationship between current and voltage can be described by a first-order differential equation
  - These are either RC or RL circuits (resistors and capacitors/inductors)
  - They can have sources or no sources
- Consider a source-free RC circuit:

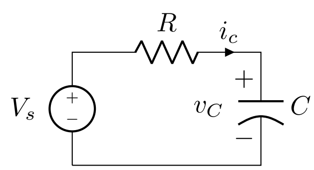


  - Suppose before time 0 the capacitor is energized to $v_c(0^-) = V_0$, and then at time 0, the switch is closed and the energizing circuit is removed
    * KVL gives: $v_c(t) + Ri_c(t) = 0 \implies v_c(t) + RC\dfrac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \implies v_c = -RC\dfrac{\mathrm{d}v_c}{\mathrm{d}t}$
    * This is a separable equation: $\displaystyle\int \frac{1}{v_c}\,\mathrm{d}v_c = \int -\frac{1}{RC}\,\mathrm{d}t \implies \ln(v_c) + K = -\frac{t}{RC}$
    * Rearranging: $v_c(t) = Ae^{-\frac{t}{RC}}$
    * Solving for initial conditions with $v_c(0) = V_0$, we obtain $v_c(t) = V_0e^{-\frac{t}{RC}}$
      - Note we can do this because a capacitor's voltage cannot change abruptly, so $v_c(0^+) = v_c(0^-) = V_0$
      - We could not have started with current because we don't know the current at 0^+
    * The current is then $i_c(t) = C\dfrac{\mathrm{d}v_c}{\mathrm{d}t} = -\dfrac{V_0C}{RC}e^{-\frac{t}{RC}} = -\dfrac{V_0}{R}e^{-\frac{t}{RC}}$ or $\dfrac{V_0}{R}e^{-\frac{t}{RC}}$ not following PSC
      - Note there is a discontinuity at time 0 as the current starts at 0 and jumps to $\dfrac{V_0}{R}$, and then decays to 0

- In a source-free RC circuit the voltage across the capacitor follows an exponential decay to 0
  - Large RC causes slower decay; small RC causes faster decay
  - Let $\tau = RC$ be the *time constant* of the RC circuit; τ characterizes how fast the voltage decays
  - τ can be found by finding the tangent at $t = 0$, and finding where the tangent intersects the horizontal axis
    * $\frac{\mathrm{d}v_c}{\mathrm{d}t}(0) = -\frac{V_0}{RC}$ so tangent is $y = V_0 - \frac{V_0 t}{RC}$; therefore when $t = \tau = RC$ the tangent intersects the time axis
  - τ has the same unit as time (seconds), so that the argument of the exponential is unitless

Lecture 25, Mar 16, 2022

Step Response of an RC Circuit

- What happen if we add an independent source to the RC circuit?
- 

  - KVL gives: $-V_s + Ri_C + v_C = 0$
    * $i_C = C\frac{\mathrm{d}v_C}{\mathrm{d}t} \implies -V_s + RC\frac{\mathrm{d}v_C}{\mathrm{d}t} + v_C(t) = 0 \implies \frac{\mathrm{d}v_C}{\mathrm{d}t} = \frac{-(v_c(t) - V_s)}{RC}$
    * This is again separable:

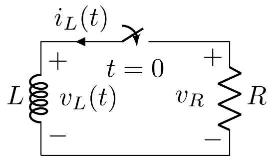
$$\begin{aligned} \int \frac{1}{v_C(t) - V_s}\, \dot{\mathrm{v}}_C &= -\int \frac{1}{RC}\,\mathrm{d}t \\ \implies \ln(v_C(t) - V_s) &= -\frac{t}{RC} + K \\ \implies v_C(t) - V_s &= Ae^{-\frac{t}{RC}} \\ \implies v_C(t) &= Ae^{-\frac{t}{RC}} + V_s \end{aligned}$$

    * Using the initial condition that $v_C(0^+) = V_0 \implies V_0 = A + V_s \implies A = V_0 - V_s$
    * Finally, $v_C(t) = V_s + (V_0 - V_s)e^{-\frac{t}{RC}}$
- As $t \to \infty$ we have $v_C(t) \to V_s$
  - At $t = 0$ there is a sharp corner as the voltage starts either increasing or decreasing and exponentially decaying to V_s
- To find the current: $i_C(t) = C\frac{\mathrm{d}v_C}{\mathrm{d}t} = \frac{V_s - V_0}{R}e^{-\frac{t}{RC}}$
  - At $t = 0$ there is a discontinuity in the current as it jumps to $\frac{V_s - V_0}{R}$; as $t \to \infty, i_C \to 0$
- In general $v_C(t) = v_C(\infty) + (v_C(0) - v_C(\infty))e^{-\frac{t}{RC}}$
- This can also be applied to any general linear circuit connected to the capacitor by finding its Thevenin equivalent
- The result can be broken down into two parts: a contribution from the initial voltage, and a part from the source: $v_C(t) = V_s(1 - e^{-\frac{t}{RC}}) + V_0 e^{-\frac{t}{RC}}$
  - The second part is exactly the behaviour of the source-free circuit
  - The first part is called the *forced response*, and the second part is the *natural response*

Lecture 26, Mar 18, 2022

Source-Free RL Circuits

- Consider a source-free RL circuit:



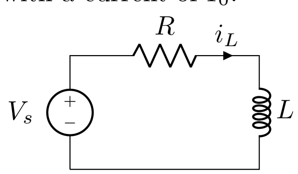
  - Initial condition: $i_L(0) = I_0$
    * Since the current of an inductor cannot change abruptly, we find the current
    * KVL: $v_L - v_R = 0 \implies L\dfrac{\mathrm{d}i_L}{\mathrm{d}t} + iR = 0$
    * Solving the differential equation: $\displaystyle\int \frac{1}{i_L}\,\mathrm{d}i_L = -\int \frac{1}{\frac{L}{R}}\,\mathrm{d}t \implies \ln(i_L(t)) + K = -\frac{t}{\frac{L}{R}}$
    * Solution is $i_L(t) = Ae^{-\frac{t}{L/R}}$
    * Using the initial condition gives $A = I_0$, giving $i_L(t) = I_0e^{-\frac{t}{L/R}} = I_0e^{-\frac{t}{\tau}}$ where $\tau = \dfrac{L}{R}$ is the time constant for an RL circuit
      - Larger time constant means slower decay
      - Similar to τ for a capacitor, the time constant can be found by the intersection of the tangent line at $t = 0$ with the time axis

    * Voltage: $v_L = L\dfrac{\mathrm{d}i_L}{\mathrm{d}t} = -RI_0e^{-\frac{t}{\tau}}$

Lecture 27, Mar 21, 2022

Step Response of an RL Circuit

- Consider an RL circuit with an voltage source that turns on at $t = 0$, with the inductor initially charged with a current of I_0:



  - KVL:

$$
\begin{aligned}
& -V_s + Ri_L(t) + L\frac{\mathrm{d}i_L}{\mathrm{d}t} = 0 \\
\implies\ & i_L(t) - \frac{V_s}{R} = -\frac{L}{R}\frac{\mathrm{d}i_L}{\mathrm{d}t} \\
\implies\ & \int \frac{1}{i_L(t) - \frac{V_s}{R}}\,\mathrm{d}i_L = \int -\frac{1}{L/R}\,\mathrm{d}t \\
\implies\ & \ln\left(i_L(t) - \frac{V_s}{R}\right) + K = -\frac{t}{L/R} \\
\implies\ & i_L(t) = \frac{V_s}{R} + Ae^{-\frac{t}{L/R}}
\end{aligned}
$$

    * Using the initial condition of $i_L(0) = I_0 \implies A = I_0 - \dfrac{V_s}{R}$
    * $i_L(t) = \dfrac{V_s}{R} + \left(I_0 - \dfrac{V_s}{R}\right)e^{-\frac{t}{L/R}} = \dfrac{V_s}{R} + \left(I_0 - \dfrac{V_s}{R}\right)e^{-\frac{t}{\tau}}$
    * $v_L(t) = L\dfrac{\mathrm{d}i_L}{\mathrm{d}t} = (V_s - RI_0)\,e^{-\frac{t}{L/R}}$

- The current starts at I_0 and exponentially decays towards $\frac{V_s}{R}$, the final value for current (at which point the inductor is a short circuit)
  - At $t = 0$ this is accompanied by a jump in the voltage
- The time constant τ is given by $\frac{L}{R}$
- As with the RC case, $i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\frac{t}{\tau}}$
  - $i_L(\infty) = \frac{V_s}{R}$ because the inductor becomes a short circuit
  - $i_L(0) = I_0$

Lecture 28, Mar 23, 2022

Sinusoids and Phasors

- Many AC sources generate sinusoidal voltages and currents (e.g. power generators, the power grid)
- Generally $v(t) = V_m \sin(\omega t + \alpha)$ for a sinusoidal voltage
  - V_m is the amplitude of the sinusoidal voltage in volts
  - ω is the angular frequency, in rad/s
  - t is time in seconds, so that ωt has units of rad
    * The period is related by $T_0 = \frac{2\pi}{\omega}$, or $\omega = \frac{2\pi}{T_0}$
    * T_0 is the *fundamental period*, the smallest possible period
  - α is the phase/phase angle/initial phase/etc in radians
    * Note sometimes α may be given in degrees, in which case you need to multiply by $\frac{\pi}{180°}$ to convert to radians
    * A signal with larger α *leads* another signal, while a smaller α *lags* another signal
    * Phase leads or lags are commonly expressed with an angle in the $-\pi$ to π range (e.g. "leading by 270°" is unconventional)
  - We also define *frequency* (as opposed to angular frequency) as $f = \frac{1}{T_0} = \frac{\omega}{2\pi}$ with units of $\mathrm{s}^{-1} = \mathrm{Hz}$
- Note the phase difference between two sinusoidal signals is $\alpha_1 - \alpha_2$ in radians, but to convert this to time we need to divide by ω
  - If two ω are different for two signals, the phase difference is undefined
  - Cosine has an additional phase offset of $+\frac{\pi}{2}$ when compared to sine
  - Adding a phase offset of 180° negates the sign
  - $\sin(\alpha \pm 180°) = -\sin(\alpha)$
  - $\cos(\alpha \pm 180°) = -\cos(\alpha)$
  - $\sin(\alpha \pm 90°) = \pm\cos(\alpha)$
  - $\cos(\alpha \pm 90°) = \mp\sin(\alpha)$
- Often sinusoidal signals are defined with a cosine function as $v(t) = V_m \cos(\omega t + \alpha)$
- Using Euler's formula, $V_m e^{j(\omega t + \alpha)} = V_m \cos(\omega t + \alpha) + jV_m \sin(\omega t + \alpha)$ (where $j^2 = -1$)
  - $v(t) = \mathrm{Re}\left(V_m e^{j(\omega t + \alpha)}\right) = \mathrm{Re}\left(V_m e^{j\alpha} e^{j\omega t}\right)$
  - For a given ω, $v(t)$ is uniquely determined by a complex number $V_m e^{j\alpha}$ (magnitude is V_m, argument is α)
    * This complex number is the *phasor* for the voltage, indicated by $\boldsymbol{V} = V_m e^{j\alpha}$
- Similarly for currents, the phasor for $i(t) = I_m \cos(\omega t + \alpha)$ is $\boldsymbol{I} = I_m e^{j\alpha}$

Lecture 29, Mar 25, 2022

Phasors (Continued)

- Phasors are often shown on a phasor diagram on a complex plane, where the length is the magnitude, and the angle is the phase offset
- Alternatively they can be expressed as $\boldsymbol{V} = V_m\angle\alpha = V_m e^{j\alpha}$
- Example: Phasor for $v(t) = V_m \cos(377t + 60°)$ is $\boldsymbol{V} = V_m e^{j60°}$; phasor for $i(t) = I_m \sin(377t + 30°)$ is $\boldsymbol{I} = I_m e^{-j60°}$
  - Note phasors are defined in terms of cosines, which is why 30° becomes −60°
  - Also note how the frequencies don't affect the phasors
- Phasors can be used to add together two sinusoids in frequency domain
- Example: Using phasors, find the sum of $i_1(t) = 4\cos(\omega t + 30°)$ and $i_2(t) = 5\sin(\omega t - 20°)$
  - $\boldsymbol{I}_1 = 4e^{j30°}, \boldsymbol{I}_2 = 5e^{-j110°}$
  - Convert phasors to rectangular format to add them: $\boldsymbol{I}_1 = 4\cos(30°) + 4i\sin(30°), \boldsymbol{I}_2 = 5\cos(-110°) + 5i\sin(-110°)$
  - $\boldsymbol{I}_1 + \boldsymbol{I}_2 = 1.754 - 2.698j = 3.218e^{-j56.98°}$

Derivatives and Integrals of Sinusoids

- $v(t) = V_m\cos(\omega t + \alpha) \implies \boldsymbol{V} = V_m e^{j\alpha}$
- $$\frac{\mathrm{d}v}{\mathrm{d}t} = -V_m\omega\sin(\omega t + \alpha) = V_m\omega\cos\left(\omega t + \alpha + \frac{\pi}{2}\right)$$
  $\implies \boldsymbol{V} = V_m\omega e^{j\left(\alpha + \frac{\pi}{2}\right)} = \left(\omega e^{j\frac{\pi}{2}}\right)\left(V_m e^{j\alpha}\right) = j\omega V_m e^{j\alpha}$
- Taking the time-domain derivative multiplies the phasor by $j\omega$
- $$\int v(t)\,\mathrm{d}t = \frac{V_m}{\omega}\sin(\omega t + \alpha) = \frac{V_m}{\omega}\cos\left(\omega t + \alpha - \frac{\pi}{2}\right) = \frac{1}{j\omega}V_m e^{j\alpha}$$
  $\implies \boldsymbol{V} = \frac{V_m}{\omega}e^{j\left(\alpha - \frac{\pi}{2}\right)} = \left(\frac{1}{\omega}e^{-j\frac{\pi}{2}}\right)\left(V_m e^{j\alpha}\right) = -\frac{j}{\omega}V_m e^{j\alpha}$
- Taking the time-domain integral multiplies the phasor by $\dfrac{1}{j\omega}$ (i.e. divide by $j\omega$)

Phasor Relations for R, L, C

- Suppose $i(t) = I_m\cos(\omega t + \theta_i) \implies \boldsymbol{I} = I_m e^{j\theta_i}$ passes through a resistor
  - Assuming PSC, by Ohm's law, $v(t) = Ri(t) = RI_m\cos(\omega t + \theta_i) \implies \boldsymbol{V} = RI_m e^{j\theta_i} \implies \boldsymbol{V} = R\boldsymbol{I}$
- $\boldsymbol{V} = R\boldsymbol{I}$ for a resistor; voltage and current are in phase
  - In the phasor diagram, a resistor only changes the length of the phasor and does not rotate it
- For an inductor: $v(t) = L\dfrac{\mathrm{d}i}{\mathrm{d}t}$ in the time domain, so in the phasor domain, $\boldsymbol{V} = j\omega L\boldsymbol{I}$
  - Since $j = e^{j90°}$, an inductor introduces a 90° phase difference (voltage leads current by 90°)
  - The magnitude is scaled by ωL
  - In the phasor diagram, the two phasors have an angle of 90° relative to each other
- For a capacitor: $i(t) = C\dfrac{\mathrm{d}v}{\mathrm{d}t}$ in the time domain, so in the phasor domain, $\boldsymbol{I} = j\omega C\boldsymbol{V}$ or $\boldsymbol{V} = \dfrac{1}{j\omega C}\boldsymbol{I}$
  - A capacitor also introduces a 90° phase difference (voltage lags current by 90°)

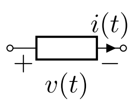
Lecture 30, Mar 28, 2022

KVL and KCL in the Phasor Domain

- Regular KVL on a loop gives
$$\begin{aligned}
&\sum_{i=1}^{n} v_i(t) = 0\\
\implies &\sum_{i=1}^{n} V_i \cos(\omega t + \alpha_i) = 0\\
\implies &\mathrm{Re}\left\{\sum_{i=1}^{n} V_i e^{j\alpha_i} e^{j\omega t}\right\} = 0\\
\implies &\mathrm{Re}\left\{\left(\sum_{i=1}^{n} V_i e^{j\alpha_i}\right) e^{j\omega t}\right\} = 0\\
\implies &\sum_{i=1}^{n} \boldsymbol{V}_i = 0
\end{aligned}$$
- Similarly for KCL on a node, $\displaystyle\sum_{i=1}^{n} \boldsymbol{I}_n = 0$
- Both KVL and KCL hold in the phasor/frequency domain

Impedance

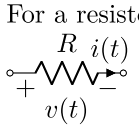
- Consider a general circuit element:



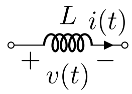
  - Suppose $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$
- *Impedance* for this element is defined as $\boldsymbol{Z} = \dfrac{\boldsymbol{V}}{\boldsymbol{I}} = \dfrac{V_m \angle \theta_v}{I_m \angle \theta_i} = \left(\dfrac{V_m}{I_m}\right) \angle(\theta_v - \theta_i)$ where $\boldsymbol{V}$ and $\boldsymbol{I}$ are the phasors for the voltage and current (assuming PSC)
  - Impedance is similar to resistance in a DC circuit
  - Unit for impedance is ohms
  - Expressed in rectangular form, $\boldsymbol{Z} = R + jX$
    * The real part R is the *resistance*
    * The imaginary part X is the *reactance*
    * Both have units of ohms
- Similar to conductance in DC circuits, for AC define the *admittance* as $\boldsymbol{Y} = \dfrac{\boldsymbol{I}}{\boldsymbol{V}} = \dfrac{1}{\boldsymbol{Z}}$
  - Admittance has the same units as conductance, Siemens or mhos
  - Expressed in rectangular form, $\boldsymbol{Y} = G + jB$, where G is the *conductance*, and B is the *susceptance*, both having units of Siemens

Impedance Relations for Passive Components

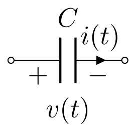
- For a resistor:



  - $\boldsymbol{Z}_R = \dfrac{\boldsymbol{V}}{\boldsymbol{I}} = \dfrac{R\boldsymbol{I}}{\boldsymbol{I}} = R$
    * The impedance of a resistor is just the resistance of that resistor
    * A resistor has no reactance; voltage and current are always in phase
- For an inductor:



  - $\boldsymbol{Z}_L = \dfrac{\boldsymbol{V}}{\boldsymbol{I}} = \dfrac{j\omega L\boldsymbol{I}}{\boldsymbol{I}} = j\omega L$
    * The impedance of an inductor is entirely imaginary, i.e. it has no resistance but ωL reactance
    * Note the impedance is dependent on frequency; when $\omega \to 0$, $\boldsymbol{Z}_L \to 0$, and when $\omega \to \infty$, $\boldsymbol{Z}_L \to \infty$
    * An inductor affects higher frequency signals more
    * Intuitively, decreasing the frequency makes the current closer to DC conditions, under which the inductor is a short circuit
- For a capacitor:



  - $\boldsymbol{Z}_C = \dfrac{\boldsymbol{V}}{\boldsymbol{I}} = \dfrac{\boldsymbol{V}}{j\omega C\boldsymbol{V}} = \dfrac{1}{j\omega C} = -\dfrac{j}{\omega C}$
    * The impedance of a capacitor is also pure imaginary; resistance is zero and reactance is $-\dfrac{1}{\omega C}$
    * When $\omega \to 0$, $\boldsymbol{Z}_C \to \infty$; when $\omega \to \infty$, $\boldsymbol{Z}_C \to 0$
    * A capacitor affects lower signal frequencies more
    * Intuitively, lower frequencies are closer to DC conditions, under which a capacitor is an open circuit

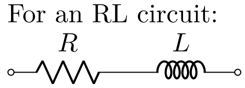
Equivalent Impedances

- Consider a number of impedance elements in series; all elements have the same $\boldsymbol{I}$ so across each there is a voltage of $\boldsymbol{V}_i = \boldsymbol{Z}_i\boldsymbol{I}$, so $\boldsymbol{V} = \boldsymbol{I}\sum\limits_{i=1}^{n}\boldsymbol{Z}_i \implies \dfrac{\boldsymbol{V}}{\boldsymbol{I}} = \sum\limits_{i=1}^{n}\boldsymbol{Z}_i$
- The equivalent impedance for a series connection is $\boldsymbol{Z}_{eq} = \sum\limits_{i=1}^{n}\boldsymbol{Z}_i$
- For a parallel connection the impedance is $\dfrac{1}{\boldsymbol{Z}_{eq}} = \sum\limits_{i=1}^{n}\dfrac{1}{\boldsymbol{Z}_i}$

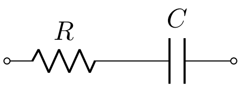
Lecture 31, Mar 30, 2022

Impedance of RL, RC, LC, and RLC circuits

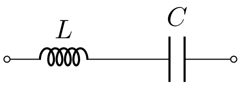
- For an RL circuit:



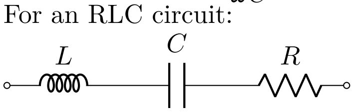
  - $\boldsymbol{Z}_{RL} = \boldsymbol{Z}_R + \boldsymbol{Z}_L = R + j\omega L$
    * The real part of the impedance for an RL circuit is the resistance of the resistor; the imaginary part is ωL, frequency times the inductance of the inductor
    * Combining R and L gives both a resistance and a reactance
    * The angle depends on both R and L; if $\boldsymbol{Z}_R \gg \boldsymbol{Z}_L$ then $\angle\boldsymbol{Z}_{RL} \to 0$; if $\boldsymbol{Z}_R \ll \boldsymbol{Z}_L$ then $\angle\boldsymbol{Z}_{RL} \to 90°$
    * The phase difference is $0 < \theta_v - \theta_i < 90°$; greater resistance leads to less phase difference, while greater inductance leads to more phase difference
    * Voltage leads current by some amount between 0° and 90°
- For an RC circuit:



- $\boldsymbol{Z}_{RC} = \boldsymbol{Z}_R + \boldsymbol{Z}_C = R - \dfrac{j}{\omega C}$
  * This time the angle is between 0° and 90° since the imaginary part (reactance) is negative
  * $\boldsymbol{Z}_R \gg \boldsymbol{Z}_C \implies \angle \boldsymbol{Z}_{RC} \to 0$ and $\boldsymbol{Z}_R \ll \boldsymbol{Z}_C \implies \angle \boldsymbol{Z}_{RC} \to -90°$
  * Voltage lags current by some amount between 0° and 90° (current leads voltage)
- For an LC circuit:



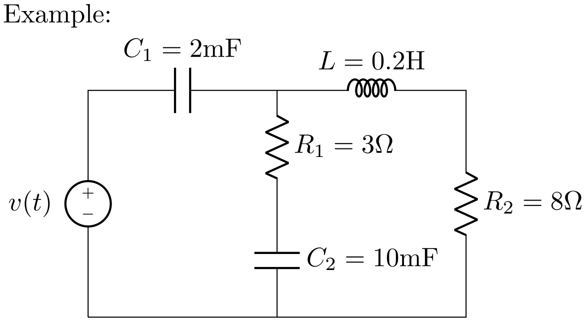
- $\boldsymbol{Z}_{LC} = \boldsymbol{Z}_L + \boldsymbol{Z}_C = j\left(\omega L - \dfrac{1}{\omega C}\right)$
  * The impedance of an LC circuit is entirely imaginary (no resistance)
  * The imaginary part can be positive or negative, depending on the relative values of the inductance and capacitance
    - $\omega L > \dfrac{1}{\omega C} \implies \operatorname{Im} \boldsymbol{Z}_{LC} > 0$, and voltage leads current by 90°
    - $\omega L < \dfrac{1}{\omega C} \implies \operatorname{Im} \boldsymbol{Z}_{LC} < 0$, and voltage lags current by 90°
- For an RLC circuit:



- $\boldsymbol{Z}_{RLC} = \boldsymbol{Z}_R + \boldsymbol{Z}_C + \boldsymbol{Z}_L = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$
  * The real part is positive, the imaginary part can be positive or negative depending on the relative values of inductance and capacitance
  * The angle is between −90° and 90°; sign follows the same pattern as for an LC circuit

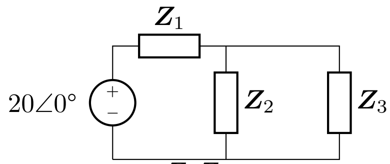
Sinusoidal Steady State Analysis

- Since all the laws and techniques (KVL, KCL, etc) still hold in the phasor domain, we can analyze AC circuits in the same way
- Convert the circuit into phasor domain (resistances, inductances, and capacitances to impedances), and use $\boldsymbol{Z} = \dfrac{\boldsymbol{V}}{\boldsymbol{I}}$ in the same way that $R = \dfrac{v}{i}$ is used in DC circuits
  - Convert the phasors back to time domain afterwards if desired
- The only difference is that complex phasors are used instead of real numbers
- Example:



- $v(t) = 20\cos(50t)V$, find $i(t)$
  * Convert the voltage to a phasor: $\boldsymbol{V} = 20\angle 0°$
  * For the 2mF capacitor, $\boldsymbol{Z}_1 = \dfrac{-j}{\omega C_1} = \dfrac{-j}{50 \cdot 2 \times 10^{-3}} = -j10\Omega$
  * For the series RC connection in the middle, $\boldsymbol{Z}_2 = R_1 - \dfrac{j}{\omega C_2} = 3 - \dfrac{j}{50 \cdot 10 \times 10^{-3}} = 3 - j2\Omega$
  * For the series RL connection, $\boldsymbol{Z}_3 = R_2 + j\omega L = 8 + j50 \cdot 0.2 = 8 + j10\Omega$

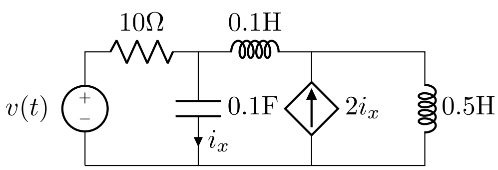
* In the phasor domain:



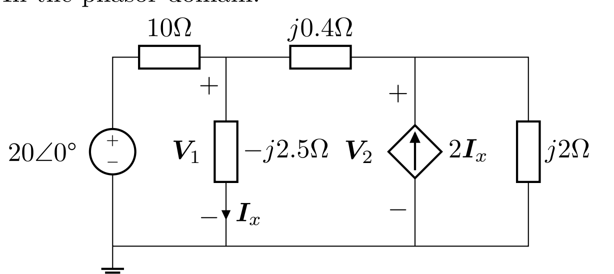
* $\boldsymbol{Z}_{eq} = \boldsymbol{Z}_1 + \dfrac{\boldsymbol{Z}_2\boldsymbol{Z}_3}{\boldsymbol{Z}_2 + \boldsymbol{Z}_3} = 3.22 - j11.07\Omega = 11.53\angle - 73.5°$
* $\boldsymbol{I} = \dfrac{\boldsymbol{V}}{\boldsymbol{Z}} = \dfrac{20\angle 0°}{11.53\angle - 73.5°} = 1.73\angle 73.5°\text{A}$
* Back to the time domain, $i(t) = 1.73\cos(50t + 73.5°)$

Lecture 32, Apr 1, 2022

Nodal and Mesh Analysis for AC Circuits

- 

  - $v(t) = 20\cos(4t) \implies \boldsymbol{V} = 20\angle 0°$
    * Impedances:
      - $\boldsymbol{Z}_R = 10\Omega$
      - $\boldsymbol{Z}_C = -\dfrac{j}{\omega C} = -\dfrac{j}{4 \cdot 0.1\text{F}} = -j2.5\Omega$
      - $\boldsymbol{Z}_{L1} = j\omega L_1 = j4 \cdot 0.1 = j0.4\Omega$
      - $\boldsymbol{Z}_{L2} = j\omega L_2 = j2\Omega$
    * In the phasor domain:



    * $\boldsymbol{I}_x = \dfrac{\boldsymbol{V}_1}{\boldsymbol{Z}_C}$
    * $\dfrac{\boldsymbol{V}_1 - 20}{\boldsymbol{Z}_R} + \dfrac{\boldsymbol{V}_1}{\boldsymbol{Z}_C} + \dfrac{\boldsymbol{V}_1 - \boldsymbol{V}_2}{\boldsymbol{Z}_{L1}} = 0$
    * $\dfrac{\boldsymbol{V}_2 - \boldsymbol{V}_1}{\boldsymbol{Z}_{L_1}} + \dfrac{\boldsymbol{V}_2}{\boldsymbol{Z}_{L2}} - 2\dfrac{\boldsymbol{V}_1}{\boldsymbol{Z}_C}$
    * Solve the system as normal, then use nodal voltages to find phasor for $\boldsymbol{I}_x = \dfrac{\boldsymbol{V}_1}{\boldsymbol{Z}_C}$, and convert to time domain

Lecture 33, Apr 4, 2022

Thevenin and Norton Equivalent Circuits for AC Circuits

- Thevenin and Norton equivalent circuits can be found for AC circuits as well
- Thevenin voltage and Norton current become the phasors for the Thevenin voltage and Norton current

- The Thevenin/Norton resistance becomes an impedance
- Thevenin voltage/resistance and Norton current can be found in the same way as in the DC case
- Source transformation also applies for AC circuits with impedances and phasors

Power in AC Circuits

- For AC circuits $P = vi$ also holds, but now v and i are time-variant, so additional complexity is involved
- Power is a function of time: $P(t) = v(t)i(t) = v(t)i(t)$
  - In the time domain, $v(t) = V_m \cos(\omega t + \theta_v), i(t) = I_m \cos(\omega t + \theta_i)$
  - Without loss of generality change the time reference so that $\theta_v = 0$
  - $P(t) = V_m I_m \cos(\omega t) \cos(\omega t - \theta)$ where $\theta = \theta_v - \theta_i$
  - Use $\cos(\alpha)\cos(\beta) = \dfrac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$
  - $P(t) = \dfrac{V_m I_m}{2}(\cos(\theta) + \cos(2\omega t - \theta))$
  - This $P(t)$ is the *instantaneous power* (in volt-amps)
- Instantaneous power is split into two parts: $\dfrac{V_m I_m}{2}\cos\theta$, the constant part, and $\dfrac{V_m I_m}{2}\cos(2\omega t - \theta)$, the time-variant term
  - Plotting power against time shows a sinusoid offset above the time axis
  - Since $\cos\theta \leq 1$ we always have $P(t) \leq V_m I_m$
  - θ shifts the curve up or down as well as left and right

Lecture 34, Apr 4, 2022

Different Types of Power in AC Circuits

- Instantaneous power: $P(t) = \dfrac{V_m I_m}{2}\cos(\theta) + \dfrac{V_m I_m}{2}\cos(2\omega t - \theta)$
- *Average power* (aka *real* or *active power*): $P_{ave} = \dfrac{V_m I_m}{2}\cos(\theta_v - \theta_i)$
  - If we integrate the instantaneous power over one period, the time-variant second term cancels out, and we're left with only the first term
  - Average power is in watts
- Using phasors, if $\begin{cases} v(t) = V_m \cos(\omega t) \\ i(t) = I_m \cos(\omega t - \theta) \end{cases} \implies \begin{cases} \boldsymbol{V} = V_m \angle 0^\circ \\ \boldsymbol{I} = I_m \angle -\theta \end{cases}$, then $\dfrac{\boldsymbol{V}\boldsymbol{I}^*}{2} = \dfrac{(V_m \angle 0^\circ)(I_m \angle \theta)}{2} = \dfrac{V_m I_m}{2}\angle\theta = \dfrac{V_m I_m}{2}\cos(\theta) + j\dfrac{\boldsymbol{V}_m \boldsymbol{I}_m}{2}\sin(\theta)$, so $P_{ave} = \operatorname{Re}\left(\dfrac{\boldsymbol{V}\boldsymbol{I}^*}{2}\right)$
- $$\begin{aligned} P(t) &= \frac{V_m I_m}{2}\cos(\theta) + \frac{V_m I_m}{2}(2\omega t - \theta) \\ &= \frac{V_m I_m}{2}\cos(\theta) + \frac{V_m I_m}{2}\cos(2\omega t)\cos(\theta) + \frac{V_m I_m}{2}\sin(2\omega t)\sin(\theta) \\ &= \frac{V_m I_m}{2}\cos(\theta)(1 + \cos(2\omega t)) + \frac{V_m I_m}{2}\sin(\theta)\sin(2\omega t) \end{aligned}$$
- *Reactive power*: $Q = \dfrac{V_m I_m}{2}\sin(\theta)$
  - Reactive power has units of volt-amp-reactive, more commonly known as VAR
- Note that from the relation derived above for average power, we have $Q = \operatorname{Im}\left(\dfrac{\boldsymbol{V}_m \boldsymbol{I}_m^*}{2}\right)$
- Instantaneous power in terms of average and reactive power: $P(t) = P_{ave}(1 + \cos(2\omega t)) + Q\sin(2\omega t)$
- Conservation holds for both active and reactive power

AC Power for R, L, and C

- For a resistor voltage and current are in phase, so $\theta_v = \theta_i \implies \theta = 0 \implies P_{ave} = \dfrac{V_m I_m}{2}, Q = 0$

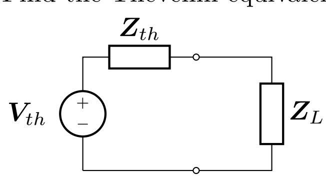
- For instantaneous power, we are left with only the first term: $P(t) = \frac{V_m I_m}{2}(1 + \cos(2\omega t))$
- Plotting this gives a sinusoid with a DC offset equal to the amplitude, i.e. the value varies between 0 and $V_m I_m$
- $P_{ave} = \text{Re}\left(\frac{\boldsymbol{V}\boldsymbol{I}^*}{2}\right) = \text{Re}\left(\frac{R|I|^2}{2}\right) = \text{Re}\left(\frac{RI_m^2}{2}\right) = \frac{1}{2}RI_m^2$
  * This is exactly like the expression for power for DC
- $Q = \text{Im}\left(\frac{\boldsymbol{V}\boldsymbol{I}^*}{2}\right) = 0$

- For an inductor $\theta_v - \theta_i = 90°$ so we're only left with $P(t) = \frac{V_m I_m}{2}\sin(2\omega t)$
  - This is a sinusoid with no DC offset
    * Every half-period, an inductor absorbs energy (positive power), and the next half-period it releases the same amount of energy back
  - $P_{ave} = 0$
  - $Q = \frac{V_m I_m}{2}\sin(90°) = \frac{V_m I_m}{2}$
  - Alternatively $Q = \text{Im}\left(\frac{j\omega L\boldsymbol{I}\boldsymbol{I}^*}{2}\right) = \frac{1}{2}\omega L I_m^2 = \frac{1}{2}X_L I_m^2$ where X_L is the reactance
- For a capacitor $\theta = -90°$ so $P(t) = -\frac{V_m I_m}{2}\sin(2\omega t)$
  - This is the same as an inductor but negated; every half-period it absorbs energy, and then in the next half-cycle it releases power back
  - $P_{ave} = 0$
  - $Q = \frac{V_m I_m}{2}\sin(-90°) = -\frac{V_m I_m}{2}$
  - Alternatively $Q = \text{Im}\left(\frac{-j\boldsymbol{I}\boldsymbol{I}^*}{\omega C}\right) = -\frac{1}{2\omega C}I_m^2 = \frac{1}{2}X_C I_m^2$ where X_C is the reactance

Lecture 35, Apr 6, 2022

Maximum Average Power

- Like maximum power transfer in DC circuits, we would like to know the impedance we should connect to an arbitrary linear AC circuit to maximize average power consumed by the impedance
- Find the Thevenin equivalent, and form the circuit:



  - $\boldsymbol{V} = \boldsymbol{Z}_L\boldsymbol{I}$
- Active power for the impedance is
$$\begin{aligned} P_L &= \text{Re}\left(\frac{\boldsymbol{V}\boldsymbol{I}^*}{2}\right) \\ &= \text{Re}\left(\frac{(R_L + jX_L)|\boldsymbol{I}|^2}{2}\right) \\ &= \frac{1}{2}R_L|\boldsymbol{I}|^2 \end{aligned}$$
  - In terms of the circuit parameters, $\boldsymbol{I} = \frac{\boldsymbol{V}_{th}}{\boldsymbol{Z}_{th} + \boldsymbol{Z}_L} = \frac{\boldsymbol{V}_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$
  - $|\boldsymbol{I}|^2 = \frac{|\boldsymbol{V}_{th}|^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$
  - So $P_L(R_L, X_L) = \frac{1}{2}R_L\frac{|\boldsymbol{V}_{th}|^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$
  - Now we have 2 parameters to optimize, R_L and X_L

- $\dfrac{\partial P_L}{\partial X_L} = R_L \dfrac{0 - 2(X_{th} + X_L)|\boldsymbol{V}_{th}|^2}{4((R_{th} + R_L)^2 + (X_{th} + X_L)^2)^2} = 0$
    - $\boldsymbol{V}_{th}$ cannot be zero, which means the only possibility is to have $X_{th} = -X_L$
- $\dfrac{\partial P_L}{\partial R_L} = \dfrac{|\boldsymbol{V}_{th}|^2((R_{th} + R_L)^2 + (X_{th} + X_L)^2) - 2(R_{th} + R_L)|\boldsymbol{V}_{th}|^2 R_L}{4((R_{th} + R_L)^2 + (X_{th} + X_L)^2)^2} = 0$
    - Again $\boldsymbol{V}_{th}$ can't be zero so we can cancel it out
    - $((R_{th} + R_L)^2 + (X_{th} + X_L)^2) - 2(R_{th} + R_L)R_L = 0$
    - Factor out $R_L + R_{th}$: $(R_{th} + R_L)(R_{th} + R_L - 2R_L) + (X_{th} + X_L) = 0$

$$\implies R_{th}^2 - R_L^2 + (X_{th} + X_L)^2 = 0$$

$$\implies R_L = \sqrt{R_{th}^2 + (X_{th} + X_L)^2}$$

    - Note we already derived above that $X_L = -X_{th}$, therefore $R_L = R_{th}$

- Without constraints, the impedance that maximizes power transfer is $\boldsymbol{Z}_L = R_{th} - X_{th}$
    - Max power is simply $\dfrac{1}{2} R_L \dfrac{|\boldsymbol{V}_{th}|^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2} = 8 \dfrac{|\boldsymbol{V}_{th}|^2}{R_{th}}$

Lecture 36, Apr 8, 2022

Root-Mean-Square (RMS) Power

- The RMS (aka *effective value*) of a periodic signal $x(t)$ with period T is given by $x_{rms} = x_{eff} = \sqrt{\dfrac{1}{T}\displaystyle\int_{t_0}^{t_0+T} x^2(t)\,\mathrm{d}t}$
- AC voltages are often expressed in RMS
- Consider $v(t) = V_m \cos(\omega t + \theta_v)$, $T = \dfrac{2\pi}{\omega}$
    - $$\begin{aligned} v_{rms} &= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,\mathrm{d}t} \\ &= \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta_v)\,\mathrm{d}t} \\ &= \sqrt{\frac{V_m^2}{T}\int_0^T \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\theta_v)\,\mathrm{d}t} \\ &= \sqrt{\frac{V_m^2}{T} \cdot \frac{1}{2}T} \\ &= \sqrt{\frac{V_m^2}{2}} \\ &= \frac{V_m}{\sqrt{2}} \end{aligned}$$
        - Note the second term in the integral cancels since we're integrating a sinusoidal function over a multiple of its period
- Similarly for currents $i_{rms} = \dfrac{I_m}{\sqrt{2}}$
- North American 110V AC outlets have an amplitude of $110\sqrt{2}$V
- RMS values are associated with the energy in the signal and applies to non-sinusoidal periodic waveforms as well

Lecture 37, Apr 11, 2022

Power In Terms of RMS Values

- Average power can be expressed in terms of the RMS voltage and current as $P_{ave} = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$
  - Using phasors $P_{ave} = \mathrm{Re}(\boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^*)$
- Likewise for reactive power $Q = V_{rms}I_{rms}\sin(\theta_v - \theta_i)$
  - Using phasors $Q = \mathrm{Im}(\boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^*)$
- For an impedance, recall for $\boldsymbol{Z} = R + jX$, average power $P_{ave} = \mathrm{Re}\left(\frac{(R+jX)|\boldsymbol{I}|^2}{2}\right) = \frac{1}{2}RI_m^2 = RI_{rms}^2$
- For reactive power, $Q = \mathrm{Im}\left(\frac{(R+jX)|\boldsymbol{I}|^2}{2}\right) = \frac{1}{2}XI_m^2 = XI_{rms}^2$

Apparent Power & Power Factor

- Active power $P_{ave} = \frac{1}{2}V_mI_m\cos(\theta_v - \theta_i) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$, can be divided into two terms, the *apparent power* $S = \frac{1}{2}V_mI_m = V_{rms}I_{rms}$ (in volt-amps), and the *power factor* $PF = \cos(\theta_v - \theta_i)$
- Consider an impedance $\boldsymbol{Z}$ with voltage $\boldsymbol{V}$ and current $\boldsymbol{I}$, $\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V_m}{I_m}\angle(\theta_v - \theta_i)$
  - Notice the power factor is the cosine of the angle of the impedance here $PF = \cos(\angle\boldsymbol{Z})$
- The power factor does not uniquely determine the impedance, since cosine is an even function, so given only a power factor we don't know if the angle of the impedance is positive or negative
  - Define a power factor as *leading* if current leads voltage, i.e. $\theta_v - \theta_i < 0$, and *lagging* if current lags voltage, i.e. $\theta_v - \theta_i > 0$
- For a resistor, $\angle\boldsymbol{Z} = 0$ so the power factor is $\cos(0) = 1$, neither lagging nor leading
  - This is the only type of impedance to have a power factor of 1
- For an inductor, $\angle\boldsymbol{Z} = 90°$ so the power factor is $\cos(90°) = 0$, and lagging since $\angle\boldsymbol{Z} > 0$
- For a capacitor, $\angle\boldsymbol{Z} = -90°$ so the power factor is also 0, but this time leading
- For an RL circuit, $\boldsymbol{Z} = R + j\omega L$ so $0° < \angle\boldsymbol{Z} < 90°$ so the power factor is between 0 and 1 lagging
- For an RC circuit, $\boldsymbol{Z} = R - \frac{j}{\omega C}$ and the power factor is between 0 and 1 leading
- For an RLC circuit, $\boldsymbol{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$ so the power factor is between 0 and 1, either leading or lagging, depending on the relative values of the inductance and capacitance

Lecture 38, Apr 11, 2022

Complex Power

- So far all of the types of power are real numbers; complex power is the only type of power that is complex
- The *complex power* for a sinusoidal AC circuit is defined as $\boldsymbol{S} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* = \boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^*$ where $\boldsymbol{V} = V_m\angle\theta_v, \boldsymbol{I} = I_m\angle\theta_i$
  - Complex power also has units of volt-amps
- We can also write complex power as $\boldsymbol{S} = \frac{1}{2}(V_m\angle\theta_v)(I_m\angle-\theta_i) = \frac{1}{2}V_mI_m\angle(\theta_v - \theta_i)$
  - Recall the apparent power $S = \frac{1}{2}V_mI_m$, so the complex power has an amplitude equal to the apparent power
- Using Euler's formula, in rectangular form $\boldsymbol{S} = V_{rms}I_{rms}\cos(\theta_v - \theta_i) + jV_{rms}I_{rms}\sin(\theta_v - \theta_i)$
- Complex power is related to average/active and reactive power by $\boldsymbol{S} = P_{ave} + jQ$
- The power factor is the cosine of the angle of the complex power

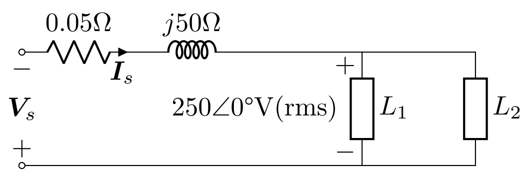
- Consider an impedance $\boldsymbol{Z} = R + jX$, complex power is $\boldsymbol{S} = \boldsymbol{V}_{rms}\boldsymbol{I}^*_{rms}$

$$\begin{aligned} &= \boldsymbol{Z}\boldsymbol{I}_{rms}\boldsymbol{I}^*_{rms} \\ &= (R + jX)|\boldsymbol{I}_{rms}|^2 \\ &= RI^2_{rms} + jXI^2_{rms} \\ &= P_{ave} + Q \end{aligned}$$

- $\boldsymbol{S}, P$ and Q form a triangle in the complex plane
- Conservation of power: the algebraic sum of all the complex powers in a circuit is zero, $\sum_{k=1}^{n} \boldsymbol{S}_k = 0$ where n is the number of circuit elements
  - Since complex power has a real and imaginary component, conservation of power holds for both average and reactive power separately

Lecture 39, Apr 12, 2022

AC Power Example

- 

  - In the circuit above, L_1 absorbs 8kW, $PF = 0.8$ leading; L_2 absorbs 20kVA, $PF = 0.6$ lagging; find the power factor of the combined load, $\boldsymbol{I}_s$, and the average power loss of the transmission line
    * L_1's power draw has units of watts, which indicates that it's an average power; L_2's power draw is in volt-amps, indicating either apparent, instantaneous or complex power, but since it's a real time-independent expression it's an apparently power
      - Complex power can only be entirely real when the power factor is 1, since in that case the angle is zero
    * Note the complex power of two combined loads is simply the sum, so for part 1, we want to find the complex power of both loads, add them and then find the angle to get the PF
    * First we need the apparent power of L_1 since we only have the average power:
      - $P_1 = V_{rms}I_{rms}PF \implies V_{rms_1}I_{rms_1} = \dfrac{8\text{kW}}{0.8} = 10\text{kW} = S_1$
      - Now we can find angle by $\theta = \cos^{-1}(0.8) = \pm 36.87°$
      - Since the power factor is leading, the angle is negative, so $\boldsymbol{S}_1 = 10000\angle - 36.87°\text{VA}$
      - Now we can calculate $\boldsymbol{S}_1 = P + jQ = 8000 + j10000\sin(-36.8°) = 8000 - j6000\text{VA}$
    * Do the same for L_2:
      - $\theta = \cos^{-1}(0.6) = \pm 53.13°$, since the power factor is lagging, this is positive
      - $P_2 = S_2 \cdot 0.6 = 12\text{kW}$
      - Combined $\boldsymbol{S}_2 = 12000 + j16000\text{VA}$
    * Adding the two loads gives a combined complex power of $20000 + j10000$VA, which has an angle of 26.56°, giving it a power factor of 0.894 lagging
    * Recall $\boldsymbol{S} = \boldsymbol{V}\boldsymbol{I}^* \implies \boldsymbol{I}^*_s = \dfrac{\boldsymbol{S}}{\boldsymbol{V}} \implies \boldsymbol{I}_s = \dfrac{\boldsymbol{S}^*}{\boldsymbol{V}^*} = \dfrac{22361\angle - 26.56°}{250\angle 0°} = 89.44\angle - 26°\text{A}$
    * To find the average power loss, we find the active power of the resistor and inductor (since the inductor has only reactive power we can ignore it)
      - $P_{loss} = R|I_s|^2 = 0.05\Omega \cdot (89.44\text{A})^2 = 400\text{W}$

Power Factor Correction

- For most practical loads, the power factors are lagging since most loads can be modelled by $\boldsymbol{Z} = R + j\omega L$; some of the power is loss through the transmission line, which is proportional to the square of the magnitude of the current phasor

- The more the voltage and current are out of phase, the more inefficient the power transmission is (large magnitude of line current which leads to high loss, but low power factor for the actual load)
- By properly choosing a capacitance in parallel with the load, we can cancel or reduce the imaginary part of $\boldsymbol{I}_{load}$, putting the voltage and current more in phase and increasing transmission efficiency ($\theta_v - \theta_i$ becomes smaller, power factor becomes bigger, which is why this method is called *power factor correction*)
- If the corrected power factor is 1, we call it *full power factor correction*; otherwise it's a *partial power factor correction*
- Example: For the previous circuit, we had a frequency of 60Hz
  - We want to choose $\boldsymbol{S}_C$ such that $\boldsymbol{S} + \boldsymbol{S}_C$ is entirely real so it has a power factor of 1
  - This means for the capacitor $Q = -j10000$, but also for a capacitor $Q = jX_C|\boldsymbol{I}_C|^2 = jX_C\left|\frac{\boldsymbol{V}}{jX_C}\right|^2 = j\frac{|\boldsymbol{V}|^2}{X_C}$
  - Therefore $\frac{|\boldsymbol{V}^2|}{X_C} = -10000 \implies X_C = -6.25\Omega$
  - From this reactance we can find the capacitance as $X_C = -\frac{1}{2\pi fC} \implies C = 424.4\text{µF}$