Lecture 8, Sep 26, 2022

Spring-Coupled Masses

- Two equal spring oscillators coupled by another spring
- $\begin{cases} m\dfrac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} = -kx_A + k(x_B - x_A) = kx_B - 2kx_A \\ m\dfrac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} = -kx_B - k(x_B - x_A) = kx_A - 2kx_B \end{cases}$
- Consider a normal mode where both masses have the same frequency, i.e. $\begin{cases} x_A(t) = A\cos(\omega t) \\ x_B)t) = B\cos(\omega t) \end{cases}$
  - Substitute into 1: $\dfrac{A}{B} = \dfrac{k}{2k - m\omega^2}$
  - Substitute into 2: $\dfrac{A}{B} = \dfrac{2k - m\omega^2}{k}$
  - Therefore $2k - m\omega^2 = \pm k \implies \omega_1 = \sqrt{\dfrac{k}{m}}, \omega_2 = \sqrt{\dfrac{3k}{m}}$
    * $\omega^2 = \dfrac{k}{m} \implies A = B$ – regular oscillation, middle spring inactive
    * $\omega^2 = \dfrac{3k}{m} \implies A = -B$ – effectively two springs in the system
- General mode would be a superposition: $\begin{cases} x_A(t) = C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t) \\ x_A(t) = C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t) \end{cases}$ where $\omega_1 = \sqrt{\dfrac{k}{m}}, \omega_2 = \sqrt{\dfrac{3k}{m}}$
- Example: Consider if $x_A(0) = A, x_B(0) = 0$:
  - $q_1 = x_A + x_B, q_2 = x_A - x_B \implies x_A = \dfrac{1}{2}(q_1 + q_2), x_B = \dfrac{1}{2}(q_1 - q_2)$
  - Consider $q_1 = C_1\cos(\omega_1 t), q_2 = C_2\cos(\omega_2 t)$; plug in $t = 0$ and solve to get $C_1 = C_2 = A$
  - $x_A = \dfrac{1}{2}(C_1\cos(\omega_1 t) + C_2\cos(\omega_2 t)) = \dfrac{1}{2}A(\cos(\omega_1 t) + \cos(\omega_2 t)) = \dfrac{1}{2}A\cos\left(\dfrac{\omega_1 + \omega_2}{2}t\right)\cos\left(\dfrac{\omega_1 - \omega_2}{2}t\right)$
    * We can think about this as a system oscillating with frequency $\dfrac{\omega_1 + \omega 2}{2}$ (i.e. the average of the two), and amplitude $\dfrac{\omega_1 - \omega_2}{2}t$
    * This is known as a beating phenomenon

Extending to More Masses

- Let $x_A = A\cos(\omega t), x_B = B\cos(\omega t)$, substitute into the equations of motion
  - Recall $\dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x$
- $\begin{cases} A(2k - m\omega^2)\cos(\omega t) = kB\cos(\omega t) \\ B(2k - m\omega^2)\cos(\omega t) = kA\cos(\omega t \end{cases} \implies \begin{cases} \dfrac{2k}{m}A - \dfrac{k}{m}B = A\omega^2 \\ \dfrac{2k}{m}B - \dfrac{k}{m}A = B\omega^2 \end{cases}$
- Write in matrix form: $\begin{bmatrix} \dfrac{2k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{2k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$
  - This is an eigenvalue problem
  - Solving us gets the same ω as before
- Example: Two equal masses m suspended from identical springs of constant k each
  - Note we don't need to worry about gravity since it's only a constant offset to the equilibrium
  - Consider a displacement down
  - $m\dfrac{\mathrm{d}^2 x_A}{\mathrm{d}t^2} = -kx_A + k(x_B - x_A) \implies \dfrac{\mathrm{d}x_A}{\mathrm{d}t} = -\dfrac{2k}{m}x_A + \dfrac{k}{m}x_B$

- $m\frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} = -k(x_B - x_A) = \frac{k}{m}x_A - \frac{k}{m}x_B$
- Let $x_A = A\cos(\omega t), x_B = B\cos(\omega t)$
- As matrix equation: $\begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix}$