Lecture 7, Sep 22, 2022

Simple Pendulum

- $-mg\sin(\theta) = L\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$
- Under a small angle approximation $-\frac{g}{L}\theta = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$
  - In this case "small" means $< 10°$
- $\theta(t) = \theta_{max}\sin(\omega t + \theta_0), \omega^2 = \frac{g}{L}, T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$
- For any arbitrary object, we can use the centre of mass point and pivot point and the moment of inertia
  - $\theta(t) = \theta_{max}\sin(\omega t + \theta_0)$
  - $\omega^2 = \frac{mgd}{I}$
  - Even though m appears in the equation, the frequency does not actually depend on mass since it cancels with the m term in I

Coupled Oscillators

- Consider two pendulums with masses m_A, m_B connected by a spring with constant k

Normal Modes

Definition

A normal mode is a mode of oscillation where every mass is oscillating at the same frequency and fixed phased relation

- Consider 2 cases:
  - Displacing them the same amount $x_A = x_B$
    - In this case the springs are unstretched; both masses oscillate in phase with the same frequency $\omega_1 = \sqrt{\frac{g}{L}}$
  - Displacing them in opposite directions $x_A = -x_B$
    - Assuming masses are the same, then $m_A\frac{\mathrm{d}^2x_A}{\mathrm{d}t^2} = -\frac{mg}{L}x_A - 2kx_A$
      - The masses are stretched by the same amount, so the total spring stretch is $2x_A$
    - Both masses oscillate with frequency $\omega_2 = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
    - The masses are out of phase
- These are the *normal modes* of oscillation
  - "Normal" since they are linearly independent

Superposition of Normal Modes

- In general $x_A \neq \pm x_B$
- Assume the angles are small so the spring is horizontal
- Restoring force on m_A is $-\frac{mg}{L}x_A - k(x_A - x_B)$, force on m_B is $-\frac{mg}{L}x_B + k(x_A - x_B)$
  - This gives us a system of coupled differential equations
- Adding the two equations shows SHM with variable $x_A + x_B$ and frequency $\omega = \sqrt{\frac{g}{L}}$
- Subtracting the two equations shows SHM with variable $x_A - x_B$ and frequency $\omega = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
  - The $\frac{2k}{m}$ term represents the coupling

- Let $x_A + x_B = q_1, x_A - x_B = q_2$, then $q_1(t) = C_1 \cos(\omega_1 t + \phi_1), q_2(t) = C_2 \cos(\omega_2 t + \phi_2)$ where $\omega_1 = \sqrt{\frac{g}{L}}, \omega_2 = \sqrt{\frac{g}{L} + \frac{2k}{m}}$
- Energy is $$E = \frac{1}{2}m\left(\frac{\mathrm{d}x_A}{\mathrm{d}t}\right)^2 + \frac{1}{2}m\left(\frac{\mathrm{d}x_B}{\mathrm{d}t}\right)^2 + \frac{1}{2}\frac{mg}{L}(x_A^2 + x_B^2) + \frac{1}{2}k(x_A - x_B)^2$$
$$= \frac{1}{4}m\left(\frac{\mathrm{d}q_1}{\mathrm{d}t}\right)^2 + \frac{1}{4}\frac{mg}{L}(q_1)^2 + \frac{1}{4}m\left(\frac{\mathrm{d}q_2}{\mathrm{d}t}\right)^2 + \frac{1}{4}\left(\frac{mg}{L} + 2k\right)(q_2)^2$$