Lecture 5, Sep 19, 2022

Driven Oscillators

- An oscillator is driven by another oscillation
- With time the frequency of the oscillator will match the driving frequency
  - Initially the movement is messy (transient response), but with time it tries to match the driver
  - There will always be a phase lag
- Maximum amplitude happens when the driving frequency matches the natural frequency of the oscillator
  - Further increasing the frequency at this point decreases the amplitude, even if the driving frequency is a multiple of the natural frequency

Undamped Forced Oscillation

- Equation of motion: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = F_0\cos(\omega t)$
  - Assume the spring moves according to $\xi(t) = \xi_0\cos(\omega t)$, then $F_0 = k\xi_0$
- The solution is $x(t) = A(\omega)\cos(\omega t - \delta)$
  - δ is the phase lag
- Substituting into the equation of motion: $A(\omega)(-\omega^2 + \omega_0^2)\cos(\delta) = \omega_0^2\xi_0, A(\omega)(-\omega^2 + \omega_0^2)\sin\delta = 0$
  - $\tan\delta = 0$, which gives us either $\delta = 0$ or $\delta = \pi$
    - With no damping, the system is either perfectly in phase or perfectly out of phase with the driver
  - $A(\omega)\left(1 - \frac{\omega^2}{\omega_0^2}\right)\cos\delta = \xi_0, A(\omega)\left(1 - \frac{\omega^2}{\omega_0^2}\right)\sin\delta = 0$
  - If $\delta = 0$: $A(\omega) = \frac{\xi_0}{1 - \frac{\omega^2}{\omega_0^2}}$ for $\omega < \omega_0$
    - Response is perfectly in phase
  - If $\delta = \pi$: $A(\omega) = \frac{-\xi_0}{1 - \frac{\omega^2}{\omega_0^2}}$ for $\omega > \omega_0$
    - Response is perfectly out of phase
- For an undamped system the amplitude goes to infinity as $\omega \to \omega_0$

Damped Forced Oscillation

- $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$
- Solution is the same $x = A(\omega)\cos(\omega t - \delta)$
- Plug back in and we get $\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$
  - As $\omega \to \omega_0, \tan\delta \to \infty \implies \delta \to \frac{\pi}{2}$
    - In resonance the response is 90 degrees out of phase
- $A(\omega) = \frac{\xi_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$
  - Amplitude is the largest for $\omega = \omega_0$, but $A(\omega)$ will never really go to infinity
  - For $\omega \to 0, A(\omega) \to \xi_0 = \frac{F_0}{k}$ (non periodic force) and $\delta \to 0$
  - For $\omega \to \omega_0, A(\omega) \to \frac{\xi_0\omega_0}{\gamma}$ and $\delta \to \frac{\pi}{2}$
  - For $\omega \to \infty, A(\omega) \to 0$ and $\delta \to \pi$
- For $\omega < \omega_0$ the oscillations are in phase, but for $\omega > \omega_0$ the oscillations become out of phase
  - This is normally a very sharp transition without damping, but with more damping it smoothes out
- Note when there is dampening, the maximum amplitude is not guaranteed when $\omega = \omega_0$
  - The largest amplitude happens when $\sqrt{(\omega_0^2 - \omega^2) + (\omega\gamma)^2}$ is minimized

  - This works out to $\omega = \omega_0\sqrt{1 - \dfrac{\gamma^2}{2\omega_0^2}}$
  - The maximum amplitude always occurs shortly before ω_0 when there is damping

Forced Oscillation Power Absorbed

- $v(t) = -v_0(\omega)\sin(\omega t - \delta)$
  - $v_0 = \dfrac{\xi_0\omega_0^2\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$
- With dampening power is dissipated at a rate of bv^2
- The instantaneous power changes, so we can integrate for the average power over a period
- $\bar{P}(\omega) = \dfrac{bv^2}{2} = \dfrac{\omega^2 F_0^2\gamma}{2m\left((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)}$
  - Note $\omega \to 0$ or $\omega \to \infty, \bar{P} \to 0$