Lecture 4, Sep 15, 2022

Damped Harmonic Oscillator Energy

- Recall in an underdamped oscillator $\omega_0^2 > \frac{\gamma^2}{4}$
- Because of the damping, energy is lost as heat
- Assuming $\frac{\gamma^2}{4} \ll \omega_0^2 \implies \omega \approx \omega_0$
  - $x(t) = A_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t)$
  - $\frac{\mathrm{d}x}{\mathrm{d}t} = -A_0 \omega_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t)$, under the approximation that $\frac{\gamma}{2\omega_0} \cos(\omega_0 t)$ is tiny
  - This leads us to $E(t) = \frac{1}{2} k A_0^2 e^{-\gamma t}$
  - Notice energy decays twice as fast as amplitude

Definition

The lifetime of a damped oscillator is $\tau = \frac{1}{\gamma}$, so $E(t) = E_0 e^{-\frac{t}{\tau}}$

- Every cycle amplitude decays by $\exp(-\gamma T)$
- $\frac{\mathrm{d}E}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}t} + kx\frac{\mathrm{d}x}{\mathrm{d}t} = v(ma + kx)$, but $ma = -kx - bv \implies ma + kx = -bv$ so $\frac{\mathrm{d}E}{\mathrm{d}t} = -bv^2$

The Q Factor

Definition

The quality factor is defined as $Q = \frac{\omega_0}{\gamma}$

- Q is the quality factor, a measurement of how good the oscillator is (how slowly it loses energy)
- For $\frac{\gamma}{2} \ll \omega_0$, every period $\frac{E(t_2)}{E(t_1)} = e^{-\gamma T}$
- With a series expansion for exp, $\frac{E(t_2 = t + T)}{E(t_1 = T)} = 1 - \gamma T$
  - $\frac{\Delta E}{E(t_1)} = -\gamma T$
  - $\frac{E(t_1) - E(t_2)}{E(t_1)} = \gamma T = \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q}$
  - This says the quality factor is the energy stored in the oscillator over the energy dissipated per radian
- The approximation $\omega \approx \omega_0$ is quite valid even for small values of Q
- $Q = \frac{1}{2}$ is the smallest Q possible, which corresponds to a critically damped oscillator
  - $\gamma = \frac{\omega_0}{Q} \implies \omega = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$

Summary

For a lightly damped harmonic oscillator ($\omega_0 \approx \omega$):

- $E(t) = \dfrac{1}{2} k A_0^2 e^{-\gamma t}$
- $A(t) = A_0 e^{-\frac{\gamma}{2} t}$
- $\dfrac{\mathrm{d}E}{\mathrm{d}t} = -bv^2$