Lecture 33 (2-17), Dec 5, 2022

Relativistic Energy

- Use work-KE theorem
- $W = \int_i^f \vec{F} \cdot \mathrm{d}\vec{s}$
- Consider 1D movement from rest at $(x_0, 0)$ to (x, t) where it has velocity u; we apply a force $F(x)$ to move the particle by $\mathrm{d}x$
  - $W = \int_{x_0}^{x} F(x)\,\mathrm{d}x$
  - To find the force, we can start with the momentum since $F = \frac{\mathrm{d}p}{\mathrm{d}t}$
  - $\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{\mathrm{d}u}{\mathrm{d}t}$
  - Also notice $u = \frac{\mathrm{d}x}{\mathrm{d}t}$ so $W = \int_{x_0}^{x} \frac{\mathrm{d}p}{\mathrm{d}t}\,\mathrm{d}x = \int_{x_0}^{x} \frac{m}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}x = \int_{x_0}^{x} \frac{m}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{\mathrm{d}u}{\mathrm{d}t} u\,\mathrm{d}t = \int_{x_0}^{x} \frac{mu}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}}\,\mathrm{d}u$
  - The final result is $W = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 = mc^2(\gamma_p - 1)$
- The kinetic energy for a particle with velocity u is $K = mc^2(\gamma_p - 1)$
  - We can check that $K = 0$ for $u = 0$
  - In the limit $\frac{u}{c} \ll 1$, we can use a Taylor series to get $\gamma_p \approx 1 + \frac{u^2}{2c^2}$ so $K = \frac{u^2}{2c^2} mc^2 = \frac{1}{2} mu^2$, which is the classic kinetic energy
- We can rearrange this to get $\gamma_p mc^2 = K + mc^2$
  - mc^2 is a form of energy that's still there even when $u = 0$ – this is the *rest energy* of the particle, which is energy that comes from the mass of the particle alone
    - We can also think of mass as a form of "potential energy"
    - Example: the binding energy of a hydrogen atom is -13.6eV; this means that the hydrogen atom is actually lighter than $m_p + m_e$, by 2.5×10^{-35} kg
  - $\gamma_p mc^2$ is then the total energy
- $E = \gamma_p mc^2 = (\gamma_p - 1)mc^2 + mc^2$
- Now can we connect p and E?
  - $E^2 = (\gamma_p mc^2)^2 = \frac{m^2c^4}{1 - \frac{u^2}{c^2}} = m^2c^4 \left(\frac{1 - \frac{u^2}{c^2} + \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}}\right) = m^2c^4 + \frac{m^2u^2c^2}{1 - \frac{u^2}{c^2}} = m^2c^4 + m^2c^4 = (pc)^2 + (mc^2)^2$
    - Therefore $E = \sqrt{(pc)^2 + (mc^2)^2}$
- We can also additionally define $\vec{\beta}_p = \frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$
  - If we use energy units of eV, then we can use momentum units of eV/c and mass units of eV/c^2
- Notice $\frac{u}{c} = \frac{pc}{E}$, so if we know p or E, we can calculate the other, and then calculate $\frac{u}{c}$ to find u