Lecture 31 (2-15), Nov 29, 2022

Length Contraction

- Consider reference frames s and s' moving with velocity v in the x direction; we try to measure the length l of a train car in s'
- If we want to do this in s, we can take the time when the start of the train passes the observer, and another time when the end passes the observer
  - This gives us a Δt and $l = v\Delta t$
- In s', $l' = v\Delta t'$
- In this case Δt is proper time so $\Delta t' = \gamma \Delta t \implies l' = \dfrac{l}{\Delta t}\Delta t' = \gamma l \implies l = \dfrac{l'}{\gamma}$
- The *proper length* is length as measured in a resting reference frame; in this case l' is the proper length
  - The observed length l is shorter, by a factor of $\dfrac{1}{\gamma}$
- $l = \dfrac{l_0}{\gamma}$ where l_0 is the proper length (length measured in the rest frame) and l is the length as observed from a moving frame

Lorentz Transformation

- The relativistic version of the Galilean transformation that addresses time dilation and length contraction
  - Connects $(x, y, z, t) \leftrightarrow (x', y', z', t')$
- Consider motion only in the x direction, between reference frames s and s' with constant velocity v
- Consider an even in s' with coordinates (x', t')
  - In the s frame $x = x' + vt$ under Galilean transformation, but x' should be length contracted so $x = \dfrac{x'}{\gamma} + vt$
  - In s' we have $\dfrac{x}{\gamma} = x' + vt'$
  - $\begin{cases} x' = \gamma(x - vt) \\ x = \gamma(x' + vt') \end{cases}$
  - $x = \gamma(\gamma x - \gamma vt) + \gamma vt' = \gamma^2(x - vt) + \gamma vt'$
  - $(1 - \gamma^2)x = -\gamma^2 vt + \gamma vt'$
  - $t' = \dfrac{1 - \gamma^2}{\gamma v}x + \gamma t = \gamma\left(t - \dfrac{v}{c^2}x\right)$
- The complete transformation: $\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{v}{c^2}x\right) \end{cases}$
  - $\begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{v}{c^2}x'\right) \end{cases}$
- What about velocity transformations?
  - Consider a particle with velocity u_x', u_y' in s', what is $\vec{u}$, as measured from s?
  - $u_x = \dfrac{\Delta x}{\Delta t}$ and from the Lorentz transformation $\dfrac{\Delta x}{\Delta t} = \dfrac{\gamma(\Delta x + v\Delta t')}{\gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} = \dfrac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v}{c^2}\frac{\Delta x'}{\Delta t'}} = \dfrac{u_x' + v}{1 + \frac{v}{c^2}u_x'}$
  - Similarly $u_y = \dfrac{\Delta y}{\Delta t} = \dfrac{\Delta y'}{\gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)} = \dfrac{u_y'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$