Lecture 3, Sep 13, 2022

Simple Harmonic Oscillator Energy

- Since the force is conservative, we have a potential, which can be calculated by the negative work done by the oscillator
- Energy is conserved in a SHO: $K + U = c = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$
  - Potential energy goes by $\cos^2$, kinetic energy goes by $\sin^2$
- Note the periodicity of energy is different (maximum kinetic/potential energy is reached twice per oscillation)

Damped Harmonic Oscillator

- Damping force at low speeds can be modelled as a drag force, proportional and opposing velocity
  - At higher speeds $F_d \propto v^2$ but this is much harder to model
- $F_{net} = -kx - bv \implies m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$
  - b is the drag coefficient, depending on the medium
  - b is also known as the damping constant (but also this name is sometimes given to γ)
- Let $\omega_0^2 = \frac{k}{m}, \gamma = \frac{b}{m} \implies \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0$
  - ω_0 is the natural frequency of the oscillation (frequency if it were undamped)
  - The damped oscillator does not oscillate at the same frequency

Light Damping (Underdamping)

- Damping is light enough that oscillations continue, losing amplitude slowly
- Predict that solution looks like $x(t) = A_0e^{-\beta t}\cos(\omega t)$ (ignore phase constant for now)
- Plugging this solution back in: $A_0e^{-\beta t}\left((2\beta\omega - \gamma\omega)\sin(\omega t) + (\beta^2 - \omega^2 - \gamma\beta + \omega^2\cos(\omega t)\right) = 0$
  - This is only zero when $2\beta\omega - \gamma\omega = 0$ and $\beta^2 - \omega^2 - \gamma\beta + \omega_0^2$
  - Solve: $\beta = \frac{\gamma}{2}, \omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$
  - Notice that the new ω is smaller, since the damping slows it down

Important

The general solution for an underdamped oscillator: $x(t) = A_0e^{-\frac{\gamma}{2}t}\cos(\omega t + \phi_0)$, where $\omega = \sqrt{\omega_0^2 + \frac{\gamma^2}{4}}$

Heavy Damping (Overdamping)

- Damping is heavy enough that no oscillations occur; system returns to equilibrium sluggishly
- We know the amplitude decays exponentially, but something else will also happen, so guess $x(t) = e^{-\beta t}f(t)$
- Plugging this back in leads to $\frac{\mathrm{d}^2f}{\mathrm{d}t^2} = \alpha^2 f, \alpha^2 = \frac{\gamma^2}{4} - \omega_0^2$
  - If $\omega_0^2 > \frac{\gamma^2}{4}$ we just have an underdamped oscillator
  - Otherwise $\alpha^2 > 0 \implies -\omega_0^2 > \frac{\gamma^2}{4}$, and $f(t) = Ae^{\alpha t} + Be^{\beta t}$

Important

The general solution for an overdamped oscillator: $x(t) = e^{-\frac{\gamma}{2}t}\left(Ae^{\alpha t} + Be^{\beta t}\right)$

Critical Damping

- When $\omega_0^2 - \frac{\gamma^2}{4} = 0$, the equation from heavy damping becomes $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = 0$
- The general solution is $f(t) = A + Bt$, so $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$
  - A has dimensions of length, B has dimensions of speed
- The behaviour is a smooth exponential decay

Important

The general solution for a critically damped oscillator: $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$